Lagrangian Dynamics 2004/05

Lecture 17: Tops: Zones, Steady Precession, Nutation, Gyroscopes, . . .

A Common Mistake: The energy equation in the form used above depends on using the conservation laws for n and L_z to write $E = E(n, L_z, \theta, \dot{\theta})$. It would however be completely wrong to "second guess" these conservation laws by trying to put them into the Lagrangian directly. For example, there is a temptation to write

$$\mathcal{L} = \frac{A}{2}(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{C}{2}n^2 - Mgl\cos\theta$$

and then obtain Lagrange's equation for θ by differentiation as usual, treating n as constant. (A similar mistake was discussed in Lecture 8, but this case is more dangerous: the result is wrong, but not obviously so.) Of course, you can always use n as shorthand for $(\dot{\psi} + \dot{\phi}\cos\theta)$ in Lagrange's equations, but then it is vital to remember that $\partial n/\partial q_i$ and $\partial n/\partial \dot{q}_i$ are nonzero in order to get the right answers. To be safe, it is probably best to write out $\mathcal{L}$ longhand in terms of the chosen generalised coordinates and velocities.

Zone Function: The fact that $\dot{\theta}$ is real requires $\dot{\theta}^2 > 0$ – compare Lecture 1. This defines for given n, L_z, E a *zone* of allowed values of the quantity

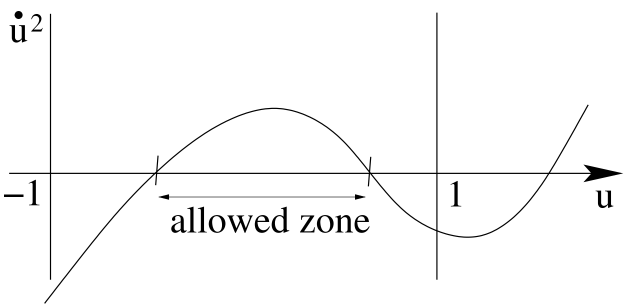
$$u \equiv \cos\theta \quad \Rightarrow \quad \dot{u} = -\dot{\theta}\,\sin\theta$$

Specifically, we can write

$$\dot{u}^2 = f(u) \equiv \left(\frac{2E - Cn^2}{A}\right)(1-u^2) - \frac{(L_z - Cnu)^2}{A^2} - \left(\frac{2Mgl}{A}\right)u(1-u^2)$$

(Check this as an exercise.)

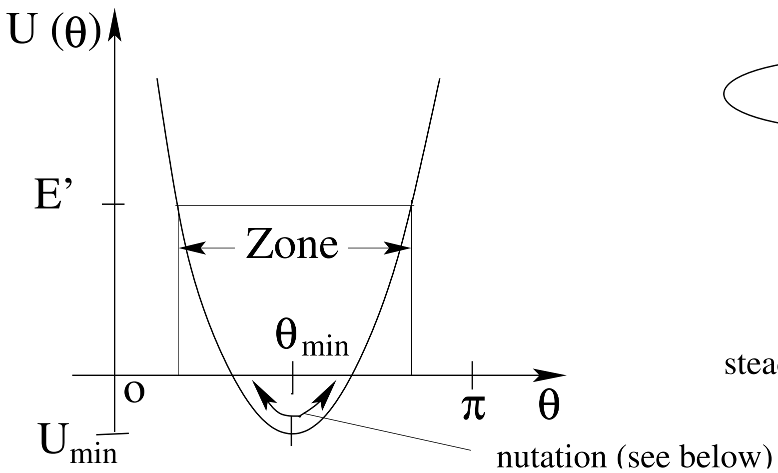
The *zone function* $f(u)$ is a cubic in u: For applications of this method, see the Problem Sheet.

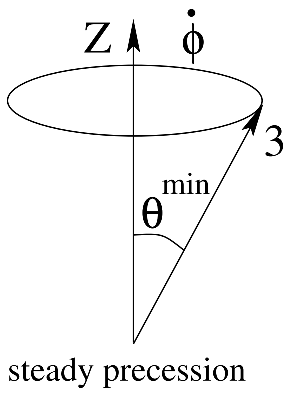


Steady Precession: From the conservation of energy we found

$$\frac{1}{2}A\dot{\theta}^2 \;=\; E - \frac{Cn^2}{2} - Mgl\cos\theta - \frac{(L_z - Cn\cos\theta)^2}{2A\sin^2\theta} \;\equiv\; E' - U(\theta)$$

where $E' = E - Cn^2/2$ is a constant and the rest of the formula defines $U(\theta)$. Remarkably, this corresponds to the energy equation for a particle of "mass" A moving with a one dimensional cartesian coordinate θ in an effective potential $U(\theta)$ with total "energy" E':





Note that $U(\theta)$ becomes very large as $\theta \to 0, \pi$, and that it is bounded from below, as in the figure. Obviously, the "potential" U is different for each value of L_z and n. This "particle analogy" gives a physically appealing alternative to the zone-function approach.

We deduce that *steady precession* (a motion with $\dot{\theta} = 0$) is possible, but that this requires $E' = U_{\rm min}(L_z, n)$ and $\theta = \theta_{\rm min}(L_z, n)$. For this case $\theta =$ constant, and since

$$\begin{aligned} p_\phi &= A\dot{\phi}\sin^2\theta + Cn\cos\theta \\ n &= \dot{\psi} + \dot{\phi}\cos\theta \end{aligned}$$

are both conserved, then $\dot{\phi}$ and $\dot{\psi}$ are also constants for this particular motion.

To find $\theta_0 \equiv \theta_{\rm min}$, we set $dU/d\theta = 0$, giving (after some algebra and substitutions):

$$\sin\theta_0 \left[\dot{\phi}^2 A\cos\theta_0 - \dot{\phi}Cn + Mgl\right] \;=\; 0 \qquad (1)$$

This result may also be obtained trivially from the Lagrange equation for θ (equation (2) of the previous lecture) by noting that $\dot{\theta} = 0$ implies $\ddot{\theta} = 0$.

Equation (1) has two solutions:

1. $\sin\theta_0 = 0$: an upright top. This is called the *sleeping top*; we return to it later.

2. $\theta_0 \neq 0$, in which case, the term in square brackets in equation (1) is zero and

   $$\dot{\phi} = \frac{Cn \pm \sqrt{C^2n^2 - 4AMgl\cos\theta_0}}{2A\cos\theta_0}$$

   so that if $\cos\theta_0 > 0$ (corresponding as usual to a top that does not hang down below its support), the existence of steady precession requires

   $$C^2n^2 \geq 4AMgl\cos\theta_0$$

   In other words, the top must be spinning fast enough if it is to show steady precession at an upward tilting angle θ_0.

Fast Top (Gyroscope): For large enough spin we can Taylor expand our equation for the angular velocity $\dot{\phi}$ of steady precession

$$\begin{aligned}\dot{\phi} &= \frac{Cn \pm \left[C^2n^2 - 4AMgl\cos\theta_0\right]^{1/2}}{2A\cos\theta_0} \\ &\simeq \frac{Cn}{2A\cos\theta_0}\left(1 \pm \left[1 - \frac{2AMgl\cos\theta_0}{C^2n^2}\right]\right)\end{aligned}$$

There are two roots:

1. **Slow Precession** (minus sign) which is due to gravity, and has

$$\dot{\phi} = \frac{Mgl}{Cn}$$

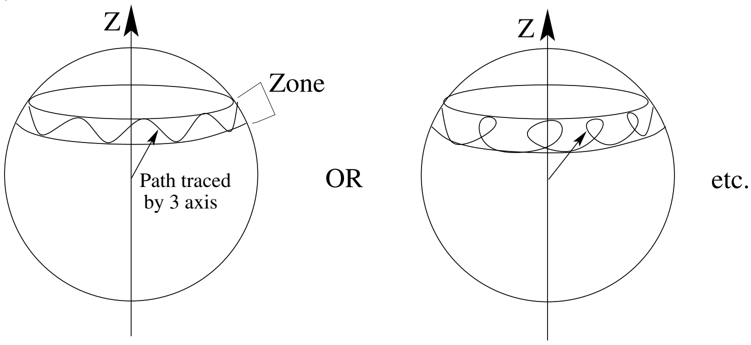
2. **Free Precession** (plus sign) where

$$\dot{\phi} = \frac{Cn}{A\cos\theta_0}$$

These limiting forms can readily be checked by other methods: for slow precession, one can balance the gravitational torque against a constant $\dot{\underline{L}}$; for free precession the result can be obtained from Euler's equations of motion (with zero torque). Note that in real gyroscopes, the free precession is rapidly damped by air friction so the slow precession is what dominates the observed behaviour.

Nutation: Suppose now we increase the energy (at fixed L_z) so that E' is slightly above U_{min}; in this case (by analogy with our 1-D particle) we have oscillations superposed on the steady precession.

The path traced by the 3 axis looks like



depending on the relative frequencies/phases of the ϕ, θ oscillations. For small amplitudes we have SHM in the θ variable with frequency

$$\Omega_{\text{nut}} = \left(U''(\theta_0)/A\right)^{1/2}$$

If the spin is large enough (the formal requirement is $Cn \gg (4AMgl\cos\theta_0)^{1/2}$) one has $\Omega_{\text{nut}} \gg \dot{\phi}$ and there are many nutational "wobbles" per precession period (*ie*, per complete circuit of the 3 axis about the vertical).

Sleeping Top: We finally return to the case $\theta_0 = 0$, that is, we look for steady motion of an upright top. Lagrange's equation for the θ motion is (see Lecture 16, page 4, item 3)

$$A\ddot{\theta} = A\dot{\phi}^2 \sin\theta\cos\theta - Cn\dot{\phi}\sin\theta + Mgl\sin\theta$$

Now expand for small θ

$$A\ddot{\theta} = (A\dot{\phi}^2 - Cn\dot{\phi} + Mgl)\theta \qquad (2)$$

Clearly, $\theta = 0$ is a solution, but is it stable?

Since the top starts off upright, $L_z = Cn$ initially. Since both L_z and Cn are conserved,

$$Cn = L_z \equiv A\dot{\phi}\sin^2\theta + Cn\cos\theta$$

So that, expanding to order θ^2,

$$Cn\theta^2/2 = A\dot{\phi}\theta^2$$

Hence the sleeping top has

$$\dot{\phi} = \frac{Cn}{2A}$$

This is the precession frequency of a sleeping top which is perturbed infinitesimally away from the vertical. Substituting back into equation (2) for $\ddot{\theta}$ gives

$$A\ddot{\theta} = \left(\frac{-C^2n^2}{4A} + Mgl\right)\theta$$

For $n^2 > 4MglA/C^2$, this is SHM and the sleeping top is stable. If the spin drops below this value, any perturbation in θ will grow exponentially and steady motion in an upright position is unstable.

The sleeping top will therefore spin happily in an upright position until such time as its spin drops below the threshold for stability. (The spin is slowly decreasing due to air friction.) Then, since there are always perturbations from somewhere, it suddenly "wakes up" and performs an elaborate motion – it goes haywire!