Homework 2

PHZ 3151

Due Friday, February 6 , 2009

1. Consider the differential equation that describes a damped harmonic oscillator with a driving force,

$$\frac{d^2y}{dt^2} + 2q\frac{dy}{dt} + \omega_0^2 y = F_D cos\Omega_D t$$

Notice that the natural frequency ω_0 is distinct from the driving frequency Ω_D.

a) Next find the particular solution to the oscillator equation above. Do not worry about the transient solution, which corresponds to the homogeneous case. Write your final answer as $y(t) = A\cos(\Omega_D t - \gamma)$, and determine expressions for the amplitude A and phase angle γ.

b) Find an expression for the driving frequency Ω_D that results in maximum amplitude A.

c) Find the expression for the instantaneous and average power delivered by the driving force.

2. Consider a simple LCR circuit driven by an AC source $V(t) = V_D cos(\Omega_D t)$. This problem should be completely review from your PHY2049 course.

a) Write the differential equation obeyed by the circuit. The equation should be in terms of the charge $Q(t)$ on the capacitor.

b) By analogy with the driven simple harmonic oscillator in problem 1, show that the particular solution is $Q(t) = Q\cos(\Omega_D t - \gamma)$. Determine expressions for the amplitude Q and the phase angle γ.

c) Determine an expression for the instantaneous and average power delivered by the AC source. Again, if you have solved problem 1, this one can be done by analogy.

d) Determine the condition for maximum average power. Notice that this resonant frequency is not the same as the frequency that results in the maximum amplitude Q.

3. Write a code using the Verlet algorithm, as discussed in class, for the damped, driven simple harmonic oscillator. You code should be able to integrate the differential equation as in problem 1,

$$\frac{d^2y}{dt^2} + 2q\frac{dy}{dt} + \omega_0^2 y = F_D cos\Omega_D t$$

In your code, compute the kinetic and potential energies. To determine if the code is behaving properly, you can monitor the total energy, which is the kinetic plus potential, as a function of time for the case where the damping and driving forces are set to zero (e.g. $\gamma = 0$ and $F_0 = 0$). Submit a plot of the kinetic, potential, and total energies as a function of time for a few periods of oscillation. The kinetic and potential energies will oscillate out of phase, and the total energy should look like a time-independent constant (i.e. a horizontal line on your plot). If the total energy is not conserved, there is either an error in the code, or you need to set the value of Δt smaller. A rule of thumb is $\Delta t \sim \frac{\pi}{50\omega_0}$ for good energy conservation.

Now treat the case with damping and a driving force. We will consider ω to be comparable to ω_0, so that the Δt you used while testing the code should work here as well. Output the integrated result for $y(t)$. In addition, output at each step the analytic result $y(t) = A\cos(\omega t - \gamma)$ determined in problem 1. You should notice that the numerical and analytical results differ at the beginning of the simulation due to the transient behavior when we abruptly turn on the driving force at $t = 0$. However, after awhile, the analytical and numerical solutions should converge, including the phase γ. If they don't, and assuming the Δt is still adequate, then either the numerical integration or the analytical solution are implemented incorrectly in the code. The analytical should be very easy to check, since you can compute by hand the amplitude!

Submit a plot of the numerical and analytical results for $y(t)$, both on the same plot for comparison. Take the case where $q = 0.01$, $F_D = 0.5$, $\Omega_D = 0.8\omega_0$, and $\omega_0 = 1$.

4. In this problem, you can slightly modify the code developed in problem 3 to study the nonlinear oscillator given by the equation,

$$\frac{d^2y}{dt^2} + 2q\frac{dy}{dt} + \omega_0^2 \sin y = F_D cos\Omega_D t$$

Follow along in the book in chapter 3 to understand the physics of the nonlinear driven pendulum. For large driving forces, chaotic behavior occurs. In particular, the behavior does not repeat itself periodically as it did in linear pendulum considered in problem 3. For this problem, submit plots like Fig. 3.6 in the book. In particular, plot θ versus time for $F_D = 0$, $F_D = 0.5$, and $F_D = 1.2$. Take $q = \frac{1}{4}$ and $\omega_0 = 1$, and $\Omega_D = \frac{2}{3}\omega_0$. Try for the timestep $\delta t = 0.4$.

For extra credit, make a plot of ω, the angular frequency, against θ for the case $F_D = 1.2$. Explain what you would expect for the non-chaotic solution in problem 3, where $\theta(t) = \theta_0 \cos(\Omega_D t - \gamma)$.