Parseval's theorem

- For a periodic function $f(x)$ defined on $-l < x < l$, we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

- The average of $[f(x)]^2$ is $\frac{1}{2l}\int_{-l}^{l}[f(x)]^2 dx$
- To obtain Parseval's theorm, use the integrals we obtained before

$$\frac{1}{2l}\int_{-l}^{l} \sin\frac{m\pi x}{l} \sin\frac{n\pi x}{l} dx = \frac{1}{2}\delta_{m,n}$$

$$\frac{1}{2l}\int_{-l}^{l} \cos\frac{m\pi x}{l} \cos\frac{n\pi x}{l} dx = \frac{1}{2}\delta_{m,n}$$

$$\frac{1}{2l}\int_{-l}^{l} \sin\frac{m\pi x}{l} \cos\frac{n\pi x}{l} dx = 0$$

Parseval's theorem continued

- Using the previous integrals, we find

$$\frac{1}{2l}\int_{-l}^{l}[f(x)]^2dx = \left(\frac{1}{2}a_0\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

- Example: Problem 5.8 and Problem 11.7
- Find the Fourier series for $f(x) = 1 + x$ defined on $-\pi < x < \pi$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(1+x)dx = 2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(1+x)\cos nxdx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(1+x)\sin nxdx = \frac{2(-1)^{n+1}}{n}$$

Example of Parseval's theorem continued

- Then Parseval's theorem states,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(1+x)^2dx = 1 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{4}{n^2} = 1 + 2\sum_{n=1}^{\infty}\frac{1}{n^2}$$

- Problem 11.8 asks us to evaluate $\sum_{n=1}^{\infty}\frac{1}{n^2}$, and from Parseval's theorem we see that,

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = -\frac{1}{2} + \frac{1}{4\pi}\int_{-\pi}^{\pi}(1+x)^2dx = \frac{\pi^2}{6}$$

- Might even use to compute π!

$$\pi = \sqrt{6}\left[\sum_{n=1}^{\infty}\frac{1}{n^2}\right]^{1/2}$$

For the fun of it... π

- Exact value of $\pi = 3.141592653589793$ (Correct to 16 digits... my computer using intrinsic functions got the digits after these incorrect)
- From serious on previous page, I got the following results:

10^4 terms: 3.141497163947214
10^5 terms: 3.141583104326456
10^6 terms: 3.141591698660508
10^7 terms: 3.141592558095902

- Correct to 7 digits for 10^7 terms, and took < 1 second to compute

Parseval's theorem for complex Fourier series

- When we average $|f(x)|^2 = f^*(x)f(x)$ over one period, we obtain $\sum_{n=-\infty}^{\infty} |c_n|^2$
- Proof in problem 3, for $f(x)$ periodic with periodicity 2π $(-\pi < x < \pi)$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

- We use the orthogonality of the functions e^{inx},

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(n-m)x} dx = \delta_{m,n}$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f^*(x) f(x) dx = \frac{1}{2\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_m^* c_n \int_{-\pi}^{\pi} e^{i(m-n)x} dx = \sum_{n=-\infty}^{\infty} c_n^* c_n$$

Another example: problem 2

• We can also average $[f(x)]^2$ using the complex series (contrast to averaging $|f(x)|^2 = f^*(x)f(x)$)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2 dx = \frac{1}{2\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_m c_n \int_{-\pi}^{\pi} e^{i(m+n)x} dx = \sum_{n=-\infty}^{\infty} c_n c_{-n}$$

• Consider the special case where $f(x)$ is real, then the expansion in complex Fourier series is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} = c_0 + \sum_{n=1}^{\infty}\left(c_n e^{inx} + c_{-n} e^{-inx}\right)$$

• Since $f(x)$ is real, the complex parts must cancel, so using the Euler formula

Problem 2 continued

$$f(x) = c_0 + \sum_{n=1}^{\infty} (c_n + c_{-n}) \cos nx + \sum_{n=1}^{\infty} (ic_n - ic_{-n}) \sin nx$$

- For the imaginary parts to go away, we require $c_{-n} = c_n^*$

$$c_n + c_{-n} = c_n + c_n^* = 2Re[c_n]$$

$$ic_n - ic_{-n} = ic_n - ic_n^* = -2Im[c_n]$$

- Then for real $f(x)$, we obtain

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} [f(x)]^2 dx = \frac{1}{2\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_m c_n \int_{-\pi}^{\pi} e^{i(m+n)x} dx = \sum_{n=-\infty}^{\infty} c_n^* c_n$$