# Laplace equation in polar coordinates

- The Laplace equation is given by

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0$$

- We have $x = r\cos\theta$, $y = r\sin\theta$, and also $r^2 = x^2 + y^2$, $\tan\theta = y/x$
- We have for the partials with respect to $x$ and $y$,

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial F}{\partial \theta}\frac{\partial \theta}{\partial x}$$

- Then $2rdr = 2xdx + 2ydy$, and $d\tan\theta = (1 + \sin^2\theta/\cos^2\theta)d\theta = -\frac{y}{x^2}dx + \frac{1}{x}dy$
- The first relation shows $\frac{\partial r}{\partial x} = x/r = \cos\theta$
- The second relation shows $\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}$

# Laplace equation in polar coordinates, continued

- So finally we get for $\frac{\partial F}{\partial x}$, and also $\frac{\partial F}{\partial y}$

$$\frac{\partial F}{\partial x} = \cos\theta \frac{\partial F}{\partial r} - \frac{\sin\theta}{r}\frac{\partial F}{\partial \theta}$$

$$\frac{\partial F}{\partial y} = \sin\theta \frac{\partial F}{\partial r} + \frac{\cos\theta}{r}\frac{\partial F}{\partial \theta}$$

- We can repeat this process, taking $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ of the above results
- Finally we obtain Laplace equation in polar coordinates,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 F}{\partial^2 \theta} = 0$$

# Another example of change of variables, Hamiltonian and Lagrangian

• Legendre transformations are also a way to change the independent variables

• In classical mechanics, we can work with $L(q, \dot{q})$ or $H(q, p)$

• Since $L(q, \dot{q})$ depends on independent variables $q$ and $\dot{q}$, we can find $dL$

• It turns out that $dL = \frac{\partial L}{\partial q} dq + \frac{\partial L}{\partial \dot{q}} d\dot{q}$ is give by,

$$dL = \dot{p} dq + p d\dot{q}$$

• Define $H = \dot{q}p - L$ (A Legendre transformation!)

$$dH = p d\dot{q} + \dot{q} dp - dL = -\dot{p} dq + \dot{q} dp$$

• Therefore $H(q, p)$ (function of independent variables $q$ and $p$

• Constructing $H(q, p)$ is the usual starting point for quantum mechanics

• Chapter 4, section 11, problem 11

# Chapter 5: Multiple integrals and applications

We often require multiple integrals in physics for obvious reasons. These will include integrals over geometric shapes including line integrals, surface integrals, and volume integrals. Sometimes symmetry and a clever change of variables can simplify multiple integrals to few dimensions. In any case, we need to explore how to use the Jacobian to write integrals in various coordinate systems. Examples include Cartesian, polar, spherical, and cylindrical coordinate systems.

Note that the first midterm tests up to the material in chapter 5!
(Lecture may go somewhat beyond chapter 5 before the test)

# Introduction

By the end of chapter 5, you should be able to...

- Understand and use double and triple integrals

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# Introduction

By the end of chapter 5, you should be able to...

- Understand and use double and triple integrals
- Make some applications of multiple integrals to physics problems
- Change variables using the Jacobian
- Work in typical coordinate systems including Cartesian, polar, cylindrical and spherical
- Use surface and volume integrals (line integrals will come in a later chapter)

# Double and triple integrals

- We might want to evaluate a volume integral, for example $z = f(x, y)$ surface to surface bounded by coordinate axes
- We have $dV = zdxdy$, which is just summing up the volume of columns of height $z$
- We have to consider the limits on $x$ and $y$, where limits might be a function of other variable
- For example, find the volume of a solid bounded by $z = 1 + y$, the vertical plane $2x + y = 2$, and the coordinate axes

$$V = \int_{x=0}^{1} \int_{y=0}^{y=2-2x} (1 + y) dydx$$

- Perform integral on $y$ first, since limits depend on $x$

$$V = \int_{x=0}^{1} \left[(2-2x) + (2-2x)^2/2\right] dx = \int_{x=0}^{1} \left[4 - 6x + 2x^2\right] dx = 5/3$$

- We could have used $x = 1 - y/2$, and integrate $x$ first, then $y$

# Another way...

- We observe that our volume is also $V = \int\int\int dxdydz$ and then we just need suitable limits
- Could use $z = 0$ on lower surface, $z = 1 + y$ on upper surface and integrate on $z$ first
- Next use $y = 0$ and $y = 2 - 2x$ on two faces and integrate on $y$
- Lastly $x = 0$ and $x = 1$ are limits on $x$

$$V = \int_{x=0}^{x=1} \int_{y=0}^{y=2-2x} \int_{z=0}^{z=1+y} dzdydx$$

- After the $z$ integral, we get same thing as before when we were summing up columns,

$$V = \int_{x=0}^{1} \int_{y=0}^{y=2-2x} (1+y)dydx = 5/3$$

# Volume integrals: coordinate systems

- We might want to integrate over a surface or a volume

$$\int\int\int F(x,y,z)dxdydz$$

- We might most easily do in another coordinate system, for example spherical coordinates

$$\int\int\int F(r,\phi,\theta)r^2 drd\phi d(\cos\theta)$$

- Of course $d\cos\theta = -\sin\theta d\theta$
- In general we have $dV$ or $d\Omega$,

$$dV = r^2 drd\phi d\cos(\theta)$$

- Or in cylindrical coordinates

$$dV = rdrd\theta dz$$

# Change of variables in the integral; Jacobian

• Element of area in Cartesian system, $dA = dxdy$

• We can see in polar coordinates, with $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$, and $\tan\theta = y/x$, that $dA = rdrd\theta$

• In three dimensions, we have a volume $dV = dxdydz$ in a Carestian system

• In a cylindrical system, we get $dV = rdrd\theta dz$

• In a spherical system, we get $dV = r^2 drd\phi d(\cos\theta)$

• We can find with simple geometry, but how can we make it systematic?

• We can define the Jacobian to make this more straightforward and automatic

# The Jacobian

- In a Cartesian system we find a volume element simply from $dV = dxdydz$
- Now assume $x \rightarrow x(u, v, w)$, $y \rightarrow y(u, v, w)$, and $z \rightarrow z(u, v, w)$
- We have in the Cartesian system $d\vec{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$
- We can then find the total differentials $dx$, $dy$, and $dz$ from

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv + \frac{\partial x}{\partial w}dw$$

$$dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv + \frac{\partial y}{\partial w}dw$$

$$dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv + \frac{\partial z}{\partial w}dw$$

# Jacobian continued

• We can define $\vec{A}$ to be along a direction such that $dv = dw = 0$, then in the Cartesian system

$$\vec{A} = \left(\hat{i}\frac{\partial x}{\partial u} + \hat{j}\frac{\partial y}{\partial u} + \hat{k}\frac{\partial z}{\partial u}\right) du$$

• Likewise $\vec{B}$ will be along a direction with $du = dw = 0$, then in the Cartesian system we see,

$$\vec{B} = \left(\hat{i}\frac{\partial x}{\partial v} + \hat{j}\frac{\partial y}{\partial v} + \hat{k}\frac{\partial z}{\partial v}\right) dv$$

• Finally $\vec{C}$ will be along a direction where $du = dv = 0$, then in the Cartesian system we see,

$$\vec{C} = \left(\hat{i}\frac{\partial x}{\partial w} + \hat{j}\frac{\partial y}{\partial w} + \hat{k}\frac{\partial z}{\partial w}\right) dw$$

# Jacobian continued

• The volume element made by these vectors is $dV = \vec{A} \cdot (\vec{B} \times \vec{C})$, which is simply the determinant

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix} dudvdw = Jdudvdw$$

• Here the determinant is the Jacobian $J$

• We have to be careful! The $J$ found above might be negative, so in general we take $|J|$

• Notice also that we can interchange rows and columns (i.e. take the transpose) and the determinant is unchanged, so

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

# Example: Volume element in cylindrical coordinates

• We know that $dV = dxdydz$ in Cartesian coordinates, and also $dV = rdrd\cos\theta dz$ in cylindrical coordinates, but let's prove it!

• We see that $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$

• We then can find $J$,

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

• So finally the element of volume $dV = Jdrd\theta dz = rdrd\theta dz$ in cylindrical coordinates

• The book proves that $dV = r^2 drd\phi d(\cos\theta)$ in Section 4, go through the proof to practice Jacobians!

# Element of area

- We might have an integral over area $dA = dxdy$, and want instead the integral in some other coordinate system
- Again assume we have $x \rightarrow x(u, v)$ and $y \rightarrow y(u, v)$
- Define vectors $\vec{B}$ and $\vec{C}$ which will lie in the $x$,$y$ plane
- For $\vec{B}$ we assume $v$ does not change

$$\vec{B} = \left( \hat{i}\frac{\partial x}{\partial u} + \hat{j}\frac{\partial y}{\partial u} \right) du$$

- For $\vec{C}$ we assume $u$ does not change

$$\vec{C} = \left( \hat{i}\frac{\partial x}{\partial v} + \hat{j}\frac{\partial y}{\partial v} \right) dv$$

- An element of area is found from $dA = |\vec{B} \times \vec{C}|$

# Element of area continued

- We find for $\vec{B} \times \vec{C}$

$$\vec{B}\times\vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} dudv = \hat{k} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} dudv = \hat{k} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} dudv$$

- We define the Jacobian $J$ as

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

- Again accounting for the fact that $J$ may be negative, we find for $dA$

$$dA = |J| dudv$$

# Example: Surface integral in polar coordinates

- We know that $dA = dxdy$, and in polar coordinates $dA = rdrd\theta$, but let's use the Jacobian to define
- We have $x = r\cos\theta$ and $y = r\sin\theta$, so we have for $J$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

- So we find as we expected for $dA$

$$dA = |J|drd\theta = rdrd\theta$$

# Elements of length

- We might need elements of arc lengths in line integrals
- In Cartesian coordinates, it is quite straightforward

$$ds^2 = dx^2 + dy^2 + dz^2$$

• To find in another system, we need $dx$ in terms of the other system, so $x \to x(u, v, w)$, etc.

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv + \frac{\partial x}{\partial w} dw$$

# Example in cylindrical coordinates

- For example, in cylindrical coordinates, we have $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$, so

$$dx = \cos\theta dr - r\sin\theta d\theta$$

$$dy = \sin\theta dr + r\cos\theta d\theta$$

$$dz = dz$$

- So we find the element of arc length in cylindrical coordinates,

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2$$

# Example in spherical coordinates

- In spherical coordinates we have $x = r\cos\phi\sin\theta$,
$y = r\sin\phi\sin\theta$, and $z = r\cos\theta$
- An element of arc length becomes,

$$ds^2 = dr^2 + r^2 d\theta + r^2 \sin^2\theta d\phi^2$$

# Surface integrals on a cylinder or a sphere

• We can see that an element $d\vec{A}$ with a magnitude equal to the area and direction normal to the surface can be found in a cylindrical system by noticing that the $\hat{z}dz$ and $\hat{\theta}ad\theta$ vectors are perpendicular, so

$$d\vec{A} = \hat{\theta}ad\theta \times \hat{z}dz = ad\theta dz\hat{r}$$

• Obviously the magnitude is $dA = ad\theta dz$
• Likewise in spherical coordinates we find $d\vec{A}$ from

$$d\vec{A} = a\hat{\phi}\sin\theta d\phi \times a\hat{\theta}d\theta = a^2\sin\theta d\phi d\theta\hat{r}$$

• In spherical coordinates the magnitude is $dA = a^2\sin\theta d\phi d\theta$

# Example: Center of mass

• We can find the center of mass coordinates $\bar{x}$, $\bar{y}$, and $\bar{z}$ defined by, in the case of a continuous mass distribution

$$\bar{x} = \frac{\int x dM}{\int dM}$$

$$\bar{y} = \frac{\int y dM}{\int dM}$$

$$\bar{z} = \frac{\int z dM}{\int dM}$$

• The significance is that when no external forces are acting on the body, the center of mass moves with a uniform velocity (or is at rest)

# More significance of the center of mass

- If there is a total (net) force $\vec{F}_{net}$, then we have

$$M\frac{d^2\bar{x}}{dt^2} = F_{net,x}$$

$$M\frac{d^2\bar{y}}{dt^2} = F_{net,y}$$

$$M\frac{d^2\bar{z}}{dt^2} = F_{net,z}$$

# Example with constant density

- With a constant density, the center of mass corresponds to the centroid of the body
- Section 3, problem 7, Find the center of mass $\bar{x}$ and $\bar{y}$ for a rectangular lamina with constant areal density $\rho = 1$ and vertices at $(0,0)$, $(0,2)$, $(3,0)$, and $(3,2)$
- The factor $dM = \rho dxdy = dxdy$ (since $\rho = 1$)
- The limits on $x$ integration are 0 and 3, and the limits on $y$ integration are 0 and 2, so

$$\bar{x} = \frac{\int_0^2 \int_0^3 x dxdy}{\int_0^2 \int_0^3 dxdy} = \frac{9}{6} = \frac{3}{2}$$

$$\bar{y} = \frac{\int_0^2 \int_0^3 y dxdy}{\int_0^2 \int_0^3 dxdy} = \frac{6}{6} = 1$$

- Not surprising, the center of mass is the centroid and is right in the middle of rectangle

# Example continued

- What if $\rho = xy$? (This is the case in problem 7)

$$\bar{x} = \frac{\int_0^2 \int_0^3 x^2 y dx dy}{\int_0^2 \int_0^3 xy dx dy} = 2$$

$$\bar{y} = \frac{\int_0^2 \int_0^3 x y^2 dx dy}{\int_0^2 \int_0^3 xy dx dy} = \frac{4}{3}$$

# Moment of inertia of a solid cylinder

• Consider a cylinder of height $h$, radius $R$, and mass $M$. Mass density is uniform.

• The volume of the cylinder is $V = \pi R^2 h$, so
$\rho = M/V = M/(\pi R^2 h)$

• Use cylindrical coordinates and determine the moment of inertia about the $z$ axis $I_z$

$$I_z = \rho \int_0^h \int_0^{2\pi} \int_0^R r^3 dr d\theta dz = \frac{M}{\pi R^2 h} \frac{2\pi R^4 h}{4} = MR^2$$

We use derivatives and various products of vectors in all areas of physics. For example, Newton's 2nd law is $\vec{F} = m\frac{d^2\vec{r}}{dt^2}$. In electricity and magnetism, we need surface and volume integrals of various fields. Fields can be scalar in some cases, but often they are vector fields like $\vec{E}(x, y, z)$ and $\vec{B}(x, y, z)$

By the end of the chapter you should be able to

- Work with various vector products including triple products

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- Work with various vector products including triple products
- Differentiate vectors

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By the end of the chapter you should be able to

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- Differentiate vectors
- Use directional derivatives and the gradient

## Chapter 6: Vector Analysis

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- Divergence and curl

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By the end of the chapter you should be able to

- Work with various vector products including triple products
- Differentiate vectors
- Use directional derivatives and the gradient
- Divergence and curl
- Line integrals
- Divergence theorem, Green theorem in plane, and Stokes theorem

# Triple products

• We have already seen that the volume of a parallelpiped from $\vec{A}$, $\vec{B}$, and $\vec{C}$ can be found

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & 0 \end{vmatrix}$$

• It is also useful to be able to find the vector product $\vec{A} \times (\vec{B} \times \vec{C})$

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$$

# Change of variables in the integral; Jacobian

• Element of area in Cartesian system, $dA = dxdy$

• We can see in polar coordinates, with $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$, and $\tan\theta = y/x$, that $dA = rdrd\theta$

• In three dimensions, we have a volume $dV = dxdydz$ in a Carestian system

• In a cylindrical system, we get $dV = rdrd\theta dz$

• In a spherical system, we get $dV = r^2 drd\phi d(\cos\theta)$

• We can find with simple geometry, but how can we make it systematic?

• We can define the Jacobian to make this more straightforward and automatic

# The Jacobian

- In a Cartesian system we find a volume element simply from $dV = dxdydz$
- Now assume $x \to x(u, v, w)$, $y \to y(u, v, w)$, and $z \to z(u, v, w)$
- We have in the Cartesian system $d\vec{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$
- We can then find the total differentials $dx$, $dy$, and $dz$ from

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv + \frac{\partial x}{\partial w}dw$$

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$$dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv + \frac{\partial z}{\partial w}dw$$

# Jacobian continued

• We can define $\vec{A}$ to be along a direction such that $dv = dw = 0$, then in the Cartesian system

$$\vec{A} = \left( \hat{i}\frac{\partial x}{\partial u} + \hat{j}\frac{\partial y}{\partial u} + \hat{k}\frac{\partial z}{\partial u} \right) du$$

• Likewise $\vec{B}$ will be along a direction with $du = dw = 0$, then in the Cartesian system we see,

$$\vec{B} = \left( \hat{i}\frac{\partial x}{\partial v} + \hat{j}\frac{\partial y}{\partial v} + \hat{k}\frac{\partial z}{\partial v} \right) dv$$

• Finally $\vec{C}$ will be along a direction where $du = dv = 0$, then in the Cartesian system we see,

$$\vec{C} = \left( \hat{i}\frac{\partial x}{\partial w} + \hat{j}\frac{\partial y}{\partial w} + \hat{k}\frac{\partial z}{\partial w} \right) dw$$

# Jacobian continued

• The volume element made by these vectors is $dV = \vec{A} \cdot (\vec{B} \times \vec{C})$, which is simply the determinant

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix} dudvdw = Jdudvdw$$

• Here the determinant is the Jacobian $J$

• We have to be careful! The $J$ found above might be negative, so in general we take $|J|$

• Notice also that we can interchange rows and columns (i.e. take the transpose) and the determinant is unchanged, so

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

# Example: Volume element in cylindrical coordinates

• We know that $dV = dxdydz$ in Cartesian coordinates, and also $dV = rdrd\cos\theta dz$ in cylindrical coordinates, but let's prove it!

• We see that $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$

• We then can find $J$,

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

• So finally the element of volume $dV = Jdrd\theta dz = rdrd\theta dz$ in cylindrical coordinates

• The book proves that $dV = r^2 drd\phi d(\cos\theta)$ in Section 4, go through the proof to practice Jacobians!

# Element of area

- We might have an integral over area $dA = dxdy$, and want instead the integral in some other coordinate system
- Again assume we have $x \longrightarrow x(u, v)$ and $y \longrightarrow y(u, v)$
- Define vectors $\vec{B}$ and $\vec{C}$ which will lie in the $x$,$y$ plane
- For $\vec{B}$ we assume $v$ does not change

$$\vec{B} = \left( \hat{i}\frac{\partial x}{\partial u} + \hat{j}\frac{\partial y}{\partial u} \right) du$$

- For $\vec{C}$ we assume $u$ does not change

$$\vec{C} = \left( \hat{i}\frac{\partial x}{\partial v} + \hat{j}\frac{\partial y}{\partial v} \right) dv$$

- An element of area is found from $dA = |\vec{B} \times \vec{C}|$

# Element of area continued

- We find for $\vec{B} \times \vec{C}$

$$\vec{B}\times\vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} dudv = \hat{k} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} dudv = \hat{k} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} dudv$$

- We define the Jacobian $J$ as

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

- Again accounting for the fact that $J$ may be negative, we find for $dA$

$$dA = |J| dudv$$

# Example: Surface integral in polar coordinates

- We know that $dA = dxdy$, and in polar coordinates $dA = rdrd\theta$, but let's use the Jacobian to define
- We have $x = r\cos\theta$ and $y = r\sin\theta$, so we have for $J$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

- So we find as we expected for $dA$

$$dA = |J| drd\theta = rdrd\theta$$

# Elements of length

- We might need elements of arc lengths in line integrals
- In Cartesian coordinates, it is quite straightforward

$$ds^2 = dx^2 + dy^2 + dz^2$$

• To find in another system, we need $dx$ in terms of the other system, so $x \to x(u, v, w)$, etc.

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv + \frac{\partial x}{\partial w} dw$$

# Example in cylindrical coordinates

- For example, in cylindrical coordinates, we have $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$, so

$$dx = \cos\theta dr - r\sin\theta d\theta$$

$$dy = \sin\theta dr + r\cos\theta d\theta$$

$$dz = dz$$

- So we find the element of arc length in cylindrical coordinates,

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2$$

# Example in spherical coordinates

- In spherical coordinates we have $x = r\cos\phi\sin\theta$,

$y = r\sin\phi\sin\theta$, and $z = r\cos\theta$

- An element of arc length becomes,

$$ds^2 = dr^2 + r^2 d\theta + r^2 \sin^2\theta d\phi^2$$

# Surface integrals on a cylinder or a sphere

- We can see that an element $d\vec{A}$ with a magnitude equal to the area and direction normal to the surface can be found in a cylindrical system by noticing that the $\hat{z}dz$ and $\hat{\theta}ad\theta$ vectors are perpendicular, so

$$d\vec{A} = \hat{\theta}ad\theta \times \hat{z}dz = ad\theta dz\hat{r}$$

- Obviously the magnitude is $dA = ad\theta dz$
- Likewise in spherical coordinates we find $d\vec{A}$ from

$$d\vec{A} = a\hat{\phi}\sin\theta d\phi \times a\hat{\theta}d\theta = a^2\sin\theta d\phi d\theta\hat{r}$$

- In spherical coordinates the magnitude is $dA = a^2\sin\theta d\phi d\theta$

## Example: Center of mass

• We can find the center of mass coordinates $\bar{x}$, $\bar{y}$, and $\bar{z}$ defined by, in the case of a continuous mass distribution

$$\bar{x} = \frac{\int x dM}{\int dM}$$

$$\bar{y} = \frac{\int y dM}{\int dM}$$

$$\bar{z} = \frac{\int z dM}{\int dM}$$

• The significance is that when no external forces are acting on the body, the center of mass moves with a uniform velocity (or is at rest)

# More significance of the center of mass

- If there is a total (net) force $\vec{F}_{net}$, then we have

$$M \frac{d^2\bar{x}}{dt^2} = F_{net,x}$$

$$M \frac{d^2\bar{y}}{dt^2} = F_{net,y}$$

$$M \frac{d^2\bar{z}}{dt^2} = F_{net,z}$$

# Example with constant density

• With a constant density, the center of mass corresponds to the centroid of the body

• Section 3, problem 7, Find the center of mass $\bar{x}$ and $\bar{y}$ for a rectangular lamina with constant areal density $\rho = 1$ and vertices at $(0,0)$, $(0,2)$, $(3,0)$, and $(3,2)$

• The factor $dM = \rho dxdy = dxdy$ (since $\rho = 1$)

• The limits on $x$ integration are 0 and 3, and the limits on $y$ integration are 0 and 2, so

$$\bar{x} = \frac{\int_0^2 \int_0^3 x dxdy}{\int_0^2 \int_0^3 dxdy} = \frac{9}{6} = \frac{3}{2}$$

$$\bar{y} = \frac{\int_0^2 \int_0^3 y dxdy}{\int_0^2 \int_0^3 dxdy} = \frac{6}{6} = 1$$

• Not surprising, the center of mass is the centroid and is right in the middle of rectangle

# Example continued

- What if $\rho = xy$? (This is the case in problem 7)

$$\bar{x} = \frac{\int_0^2 \int_0^3 x^2 y dx dy}{\int_0^2 \int_0^3 xy dx dy} = 2$$

$$\bar{y} = \frac{\int_0^2 \int_0^3 xy^2 dx dy}{\int_0^2 \int_0^3 xy dx dy} = \frac{4}{3}$$

# Moment of inertia of a solid cylinder

• Consider a cylinder of height $h$, radius $R$, and mass $M$. Mass density is uniform.
• The volume of the cylinder is $V = \pi R^2 h$, so
$\rho = M/V = M/(\pi R^2 h)$
• Use cylindrical coordinates and determine the moment of inertia about the $z$ axis $I_z$

$$I_z = \rho \int_0^h \int_0^{2\pi} \int_0^R r^3 dr d\theta dz = \frac{M}{\pi R^2 h} \frac{2\pi R^4 h}{4} = MR^2$$

We use derivatives and various products of vectors in all areas of physics. For example, Newton's 2nd law is $\vec{F} = m\frac{d^2\vec{r}}{dt^2}$. In electricity and magnetism, we need surface and volume integrals of various fields. Fields can be scalar in some cases, but often they are vector fields like $\vec{E}(x, y, z)$ and $\vec{B}(x, y, z)$
By the end of the chapter you should be able to

- Work with various vector products including triple products

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# Chapter 6: Vector Analysis

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- Use directional derivatives and the gradient
- Divergence and curl
- Line integrals
- Divergence theorem, Green theorem in plane, and Stokes theorem

# Triple products

• We have already seen that the volume of a parallelpiped from $\vec{A}$, $\vec{B}$, and $\vec{C}$ can be found

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & 0 \end{vmatrix}$$

• It is also useful to be able to find the vector product $\vec{A} \times (\vec{B} \times \vec{C})$

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$$