Chapter 2: Complex numbers

Complex numbers are commonplace in physics and engineering. In particular, complex numbers enable us to simplify equations and/or more easily find solutions to equations. We will explore the damped, driven simple-harmonic oscillator as an example of the use of complex numbers.

By the end of this chapter you should be able to...

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By the end of this chapter you should be able to...

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- Use complex algebra
- Complex infinite series
- Determine functions of complex numbers

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- Define and use hyperbolic functions
- Use logarithms
- Do all of the above with complex numbers!
- Solve harmonic oscillator and driven-damped oscillator

Examples using Euler's formula

• Express $z = 2e^{\frac{\pi i}{4}}$ in the form $z = x + iy$

From Euler's formula,

$$2e^{\frac{i\pi}{4}} = 2\cos\frac{\pi}{4} + 2i\sin\frac{\pi}{4} = \sqrt{2} + i\sqrt{2}$$

• Express $z = \left(\frac{i\sqrt{2}}{1+i}\right)^6$ in the form $z = x + iy$

Using Euler's formula, $i = e^{\frac{i\pi}{2}}$ and $1 + i = \sqrt{2}e^{\frac{\pi i}{4}}$, then we see

$$\left(\frac{i\sqrt{2}}{1+i}\right)^6 = \left(\frac{e^{\frac{i\pi}{2}}}{e^{\frac{i\pi}{4}}}\right)^6 = \left(e^{\frac{i\pi}{4}}\right)^6 = e^{\frac{3\pi i}{2}}$$

Then we use Euler's formula,

$$e^{\frac{3\pi i}{2}} = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i$$

Powers and roots of complex numbers

We can start with the form $z = re^{i\theta}$, then to take to the nth power,

$$z^n = \left(re^{i\theta}\right)^n = r^n e^{in\theta}$$

Likewise, if we want the nth root of z,

$$z^{1/n} = \left(re^{i\theta}\right)^{1/n} = r^{1/n} e^{i\theta/n}$$

We used this in the second example in the last slide

Examples of roots of complex numbers

• For $z = -8$, determine $z^{1/3} = (-8)^{1/3}$ in the form $x + iy$

For $z = -8$, we can see $r = 8$ and $\theta = \pi$, so $z = 8e^{i\pi}$, and then

$$z^{1/3} = (-8)^{1/3} = \left(8e^{i\pi}\right)^{1/3} = 2e^{i\pi/3}$$

Then we use Euler's formula,

$$2e^{i\pi/3} = 2\cos\pi/3 + 2i\sin\pi/3 = 1 + i\sqrt{3}$$

This can be easily checked without invoking Euler's formula,

$$\left(1 + i\sqrt{3}\right)^3 = -8$$

Exponential and trigonometric functions

Euler's formula can be used to find representations of $\cos\theta$ and $\sin\theta$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{1}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{2}$$

Instead of just real θ, this also applies for complex z,

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \tag{3}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \tag{4}$$

Example of complex exponentials for integration

Complex exponentials are useful for integrating products of sin and cos functions. For example

- Solve the integral $\int_{-\pi}^{\pi} \cos 2x \cos 3x dx$.

First we make note that $cos2x = \frac{e^{2ix}+e^{-2ix}}{2}$ and $cos3x = \frac{e^{3ix}+e^{-3ix}}{2}$

$$\int_{-\pi}^{\pi} \cos 2x \cos 3x dx = \frac{1}{4} \int_{-\pi}^{\pi} \left(e^{5ix} + e^{ix} + e^{-ix} + e^{-5ix} \right) dx$$

This integral is easy, and we get

$$\frac{1}{4} \left[\left(\frac{e^{5ix} - e^{-5ix}}{5i} \right) + \left(\frac{e^{ix} - e^{-ix}}{i} \right) \right]_{-\pi}^{\pi} = \left[\frac{1}{10} \sin 5x + \frac{1}{2} \sin x \right]_{-\pi}^{\pi} = 0$$

The complex exponential form is also useful in *differential equations*.

Hyperbolic functions

If we start with our representations of cos and sin as complex exponentials, then consider pure imaginary argument (e.g. $z = iy$)

$$\sin iy = i\frac{e^{y} - e^{-y}}{2} \tag{5}$$

$$\cos iy = \frac{e^{y} + e^{-y}}{2} \tag{6}$$

This provides us with definitions for the *hyperbolic functions*, $\sinh y = -i \sin iy$ and $\cosh y = \cos iy$. More generally for any z,

$$\sinh z = \frac{e^{z} - e^{-z}}{2} \tag{7}$$

$$\cosh z = \frac{e^{z} + e^{-z}}{2} \tag{8}$$

Also $\tanh z = \frac{\sinh z}{\cosh z}$, $\coth z = \frac{\cosh z}{\sinh z}$,etc.

Example with hyperbolic functions

- Write $\sinh\left(\ln 2 + \frac{i\pi}{3}\right)$ in $x + iy$ form

We use the representation of sinh in terms of exponentials,

$$sinh\left(\ln 2 + \frac{i\pi}{3}\right) = \frac{e^{(\ln 2 + i\pi/3)} - e^{-(\ln 2 + i\pi/3)}}{2} = \frac{2e^{i\pi/3} - (1/2)e^{-i\pi/3}}{2}$$

Then using Euler's formula for the complex exponentials, we get

$$sinh\left(\ln 2 + \frac{i\pi}{3}\right) = \frac{3}{8} + \frac{5\sqrt{3}}{8}i$$

Logarithms of complex numbers

- It is possible to take logarithms of negative or even complex numbers

$$\ln\left(re^{i\theta}\right) = \ln r + i(\theta \pm 2n\pi) \tag{9}$$

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- If $z = e^w$ then $w = \ln z$ where z is complex
- $w = \ln z = \ln(re^{i\theta}) = \ln r + i\theta$
- Since we can add $2n\pi$ to θ, n integer, and get same result, we have most generally:

$$\ln\left(re^{i\theta}\right) = \ln r + i(\theta \pm 2n\pi) \tag{9}$$

Complex roots and complex powers

We can take a complex number a to a complex power b! We can often evaluate using,

$$a^b = e^{b \ln a} \tag{10}$$

- For example, evaluate i^i in the form $x + iy$

$$i^i = e^{i \ln i}$$

Then using $i = e^{i\pi/2} e^{\pm 2n\pi i}$ (from Euler's formula), we see $i \ln i = -\pi/2 \pm 2n\pi$, and finally,

$$i^i = e^{-\pi/2 \pm 2n\pi}$$

While there are an infinite number of answers, note that they are all real!

Example of a complex number and a real root

- Evaluate $i^{1/2}$ in the form $x + iy$.

$$i^{1/2} = e^{(1/2)\ln i} = e^{(1/2)\ln\left(e^{i\pi/2 \pm 2n\pi i}\right)} = e^{i\pi/4 \pm in\pi}$$

Since $e^{in\pi} = 1$ for even n and $e^{in\pi} = -1$ for odd n, we have two answers,

$$i^{1/2} = \pm e^{i\pi/4} = \pm \frac{1+i}{\sqrt{2}}$$

Not surprising that the square root gives two possible results, as it does for real numbers.

- Check directly this result,

$$i^{1/2} i^{1/2} = \left[\frac{1+i}{\sqrt{2}}\right]\left[\frac{1+i}{\sqrt{2}}\right] = i$$

- For $w = \cos z$, we have $z = \arccos w = \cos^{-1} w$

It is convenient to use the forms,

$$w = \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$w = \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Inverse trigonometric and hyperbolic functions

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- If z is complex, w does not have to be between -1 and $+1$

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Example of $\cos z$, $\sin z$ with complex z

- Find $z = \arccos(i\sqrt{8})$ in the form $x + iy$

We start with $z = \arccos(i\sqrt{8})$ and write equivalently $\cos z = i\sqrt{8}$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = i\sqrt{8}$$

For simplicity take $u = e^{iz}$ then we can write,

$$\frac{u + u^{-1}}{2} = i\sqrt{8}$$

Which gives the quadratic equation,

$$u^2 - 4i\sqrt{2}u + 1 = 0$$

This has the roots $u = (2\sqrt{2} \pm 3)i$, so $iz = \ln\left[\left(2\sqrt{2} \pm 3\right) i\right]$... complete in the homework!

Chapter 3: Linear algebra

Problems in physics often lead to a set of linear equations. In solving linear equations is often convenient to use matrices and vectors. Matrices and vectors also occur frequently in the representation of states and linear operators in quantum mechanics. Determining the quantum states of a system can be reduced to solving an eigenvalue equation. Another example is coordinate transformations, which occurs in, for example, relativity and group theory, which is essential in particle physics but also crystallography amongst other areas. The vibrations of molecules and crystals can also be understood by solving large sets of linear equations. It's hard to overemphasize the importance of this subject!

Goals:

By the end of this chapter you should be able to:

- Represent a set of linear equations with matrices

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By the end of this chapter you should be able to:

- Represent a set of linear equations with matrices
- Use elementary row reduction to solve a matrix equation

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- Understand vector spaces, linear dependence/independence

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- Get some basic applications of matrix diagonalization (some physics here!)
- Learn some fundamentals of group theory
- Learn about inner products, Dirac notation (used frequently in quantum mechanics!)

Elementary row operations

Consider a set of linear equations, for example:

$$x - 2y = 13$$

$$-4x + y = 17$$

This is easily solved. From the first equation, $x = 2y + 13$. Substitute into the second equation,

$$-4x + y = -4(2y + 13) + y = -7y - 52 = 17$$

We see $y = -69/7$ and $x = 2y + 13 = -(2)(69)/7 + 13 = -47/7$

Check it!

Another approach: matrices

$$x - 2y = 13$$
$$-4x + y = 17$$

Matrix of the coefficients and two 2×1 matrices (vectors),

$$M = \begin{pmatrix} 1 & -2 \\ -4 & 1 \end{pmatrix}$$

$$r = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$k = \begin{pmatrix} 13 \\ 17 \end{pmatrix}$$

$$\sum_{j=1}^{2} M_{ij} r_j = k_i$$

For M_{ij} the i is the row, and j is the column.

The augmented matrix

To make row reduction easy, we can make this even simpler:

$$x - 2y = 13$$

$$-4x + y = 17$$

As an augmented matrix then,

$$\begin{pmatrix} 1 & -2 & 13 \\ -4 & 1 & 17 \end{pmatrix}$$

We will use *row reduction* to solve the equations

The basic rules:

We can:

- Interchange rows

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- Multiply or divide a row by a nonzero constant

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We can:

- Interchange rows
- Multiply or divide a row by a nonzero constant
- Add or subtract one row from another

Example of row reduction:

$$\begin{pmatrix} 1 & -2 & 13 \\ -4 & 1 & 17 \end{pmatrix}$$

1. Multiply first row by 4, add to second row

$$\begin{pmatrix} 1 & -2 & 13 \\ 0 & -7 & 69 \end{pmatrix}$$

2. Solve $y = -69/7$
3. From first row we find $x = 13 - (2)(69)/7 = -47/7$

Same as before! The operations we did were equivalent.

Inconsistency, linear independence

Is there always a solution? No!

Equations may be inconsistent, which suggests a mistake

For two equations and two unknowns, can you imagine two lines in the x-y plane that represent a set of linear equations with no solution? The question is whether the equations are *linearly independent*. We will return to this later.

Linearly dependent equations can also lead to infinite numbers of solutions!

Rank of a matrix

The rank of a matrix is the number of nonzero rows.

Consider the rank of M (coefficient matrix), A (augmented matrix), and n the number of unknowns after row reduction:

- rank $M =$ rank $A =$ n, one unique solution

In the last case, we can find R unknowns in terms of the $n - R$ unknowns. Might need another way to constrain solution.

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- rank $M <$ rank A, equations inconsistent

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- rank $M =$ rank $A =$ n, one unique solution
- rank $M <$ rank A, equations inconsistent
- rank $M =$ rank $A = R < n$, infinite number of solutions!

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Determinants

Determinants of a 2×2 matrix are easily evaluated. Here we define it:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$detA = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For bigger matrices, it is helpful to define the *minor* M_{ij} and *cofactor* $(-1)^{i+j} M_{ij}$.

Determinants of bigger matrices:

Write our determinant in the notation:

$$detA = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Take the element a_{11}. Its minor M_{11} is found from the smaller matrix

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$$

The cofactor of of a_{ij} is $(-1)^{i+j}M_{ij}$, so the cofactor of a_{11} is $(-1)^2 M_{11} = M_{11}$.

Determinants of bigger matrices:

For the 3×3 matrix in the last slide, we take a row or column, and multiply the elements by their cofactors.

$$detA = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

For example, we use the first row

$$detA = a_{11}(-1)^2 M_{11} + a_{12}(-1)^3 M_{12} + a_{13}(-1)^4 M_{13}$$

Could use *any* row or column and get the same result!
We get here,

$$detA = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Cramer's Rule: A use for determinants!

$$a_1 x + b_1 y = c_1$$

$$a_2 x + b_2 y = c_2$$

Using row-reduction, we find

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Some useful facts about determinants:

- If we multiply a column or row by k, then the determinant is multiplied by k.

- For the transpose A^T, we exchange rows and columns

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- Determinant is unchanged if we add rows or columns, and also if we take the transpose

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Example: Cramer's rule

Problem 17, section 3:
Use Cramer's rule to find x and t from the Lorentz equations,

$$\gamma x - \gamma v t = x'$$

$$-\frac{\gamma v}{c^2} x + \gamma t = t'$$

Write in matrix form,

$$\begin{pmatrix} \gamma & -\gamma v \\ -\frac{\gamma v}{c^2} & \gamma \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} x' \\ t' \end{pmatrix}$$

We need

$$D = \begin{vmatrix} \gamma & -\gamma v \\ -\frac{\gamma v}{c^2} & \gamma \end{vmatrix} = \gamma^2 - \gamma^2 v^2 / c^2 = 1$$

Then we find from Cramer's rule,

$$x = \frac{\begin{vmatrix} x' & -\gamma v \\ t' & \gamma \end{vmatrix}}{D} = \gamma x' + \gamma v t'$$

$$t = \frac{\begin{vmatrix} \gamma & x' \\ -\frac{\gamma v}{c^2} & t' \end{vmatrix}}{D} = \gamma t' + \frac{\gamma v}{c^2} x'$$