Solving differential equations with Fourier transforms

• Consider a damped simple harmonic oscillator with damping γ and natural frequency ω_0 and driving force $f(t)$

$$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + \omega_0^2 y = f(t)$$

• At $t = 0$ the system is at equilibrium $y = 0$ and at rest so $\frac{dy}{dt} = 0$
• We subject the system to an force acting at $t = t'$,
$f(t) = \delta(t - t')$, with $t' > 0$
• We take $y(t) = \int_{-\infty}^{\infty} g(\omega) e^{i\omega t} d\omega$ and $f(t) = \int_{-\infty}^{\infty} f(\omega) e^{i\omega t} d\omega$

Example continued

- Substitute into the differential equation and we find

$$\left[\omega_0^2 - \omega^2 + 2ib\omega\right] g(\omega) = f(\omega)$$

- We find also $f(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(t-t')e^{-i\omega t}dt = \frac{1}{2\pi}e^{-i\omega t'}$
- We find a relationship between the $g(\omega)$ and $f(\omega)$, and then we can write for the response $g(\omega)$

$$g(\omega) = \frac{1}{2\pi}\frac{e^{-i\omega t'}}{\omega_0^2 - \omega^2 + 2ib\omega}$$

- Then with $y(t) = 0$ for $t < t'$, we get $y(t)$ for $t > t'$

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega(t-t')}}{\omega_0^2 - \omega^2 + 2ib\omega} d\omega$$

Example continued

• The integral is hard to do (we might get to later), but the point is we have reduced the problem to doing an integral

• Assume $b < \omega_0$, then we find for $y(t)$ with $t > t'$,

$$y(t) = e^{-b(t-t')} \frac{\sin\left[\omega'(t-t')\right]}{\omega'}$$

where $\omega' = \sqrt{\omega_0^2 - b^2}$ and $y(t) = 0$ for $t < t'$

• You can convince yourself that this is consistent with the $b = 0$ case described in the book (see Eq. 12.5 in chapter 8)

Green functions: An introduction

• We can use as an example the damped simple harmonic oscillator subject to a driving force $f(t)$ (The book example corresponds to $\gamma = 0$)

$$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + \omega_0^2 y = f(t)$$

• Now that we know the properties of the Dirac delta function, we notice that $f(t) = \int_{-\infty}^{\infty} f(t')\delta(t - t')dt'$

• This gives a hint that we can treat $f(t)$ as a sequence of delta-function impulses

$$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + \omega_0^2 y = f(t)$$

• Let's say $f(t)$ is zero for $t < 0$, and also $y(t) = 0$ for $t < 0$, and then we turn on the driving force $f(t)$

• Using our insight, and the principle of superposition, we assume that the response ($y(t)$) depends on the entire history of the force $f(t')$ from $0 < t' < t$,

$$y(t) = \int_0^t G(t,t')f(t')dt'$$

Green function for damped oscillator

- Substitute this into the equation of motion

$$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + \omega_0^2 y = f(t)$$

- Use $y(t) = \int_0^t G(t,t')f(t')dt'$ and $f(t) = \int_0^\infty f(t')\delta(t'-t)dt'$

$$\int_0^t f(t')\left[\left(\frac{d^2}{dt^2} + 2b\frac{d}{dt} + \omega_0^2\right)G(t,t')\right]dt' = \int_0^\infty f(t')\delta(t'-t)dt'$$

continued

• We see that the *Green function* $G(t, t')$ solves the differential equation,

$$\left(\frac{d^2}{dt^2} + 2b\frac{d}{dt} + \omega_0^2\right) G(t, t') = \delta(t' - t)$$

• Note also that $G(t, t') = 0$ for $t < t'$
• We already solved that! It was just the response $y(t)$ due to a δ-function impulse, with $\omega' = \sqrt{\omega_0^2 - b^2}$

$$G(t, t') = e^{-b(t-t')}\frac{\sin\left[\omega'(t - t')\right]}{\omega'}$$

• Notice that the response only depends on $t - t'$, as we expect
• This was for the underdamped case ($b < \omega_0$), and would not work for critical or overdamped cases!

Last one! Green function for damped oscillator

• Finally we can write the solution $y(t)$ for *any* driving force $f(t)$ turned on at $t = 0$, for the damped oscillator in the underdamped regime,

$$y(t) = \int_0^t G(t,t')f(t')dt' = \int_0^t e^{-b(t-t')} \frac{\sin\left[\omega'(t-t')\right]}{\omega'} f(t')dt'$$

Green functions continued

• Quite powerful! As long as differential equation is linear, we can find the Green (response) function which completely solves any problem

• Another example: Electrostatics

• We know that the electrostatic potential $\phi(\vec{r})$ due to a continuous charge distribution $\rho(\vec{r}')$ is simply additive

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} d^3\vec{r}'$$

• Because of this, Gauss' Law is a linear differential equation,

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$

• Then, since $E = -\vec{\nabla}\phi$, we have

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0}$$

Green function for electrostatics

• We will see that $G(\vec{r},\vec{r}') = \frac{1}{4\pi\epsilon_0}\frac{1}{|\vec{r}-\vec{r}'|}$

• First, take note that $\rho(\vec{r}) = \int \rho(\vec{r'})\delta(\vec{r}-\vec{r}')d^3\vec{r}'$

• It might be more clear if we note that $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{r}' = x'\hat{i} + y'\hat{j} + z'\hat{k}$, and then

$$\rho(\vec{r}) = \int\int\int \rho(\vec{r'})\delta(x-x')\delta(y-y')\delta(z-z')dx'dy'dz'$$

• Next we use that the potential $\phi(\vec{r})$ is found just by adding up the contributions due to each part of $\rho(\vec{r}')$, so

$$\phi(\vec{r}) = \int G(\vec{r},\vec{r}')\rho(\vec{r}')d^3\vec{r}'$$

Green function for electrostatics

- Substitute into the Gauss Law expression $\nabla^2\phi = -\frac{\rho}{\epsilon_0}$

$$\int \nabla^2 G(\vec{r},\vec{r}')\rho(\vec{r}')d^3\vec{r}' = -\frac{1}{\epsilon_0}\int \rho(\vec{r'})\delta(\vec{r}-\vec{r}')d^3\vec{r}'$$

- Noting that the ∇^2 is with respect to $\vec{r}$ (and not $\vec{r}'$, we get the equation for the Green function

$$\nabla^2 G(\vec{r},\vec{r}') = -\frac{1}{\epsilon_0}\delta(\vec{r}-\vec{r}')$$

- Then $G(\vec{r},\vec{r}')$ is just the potential at $\vec{r}$ due to a unit charge located at $\vec{r}'$
- Since we know Coulomb's Law, we can see right away that $G(\vec{r},\vec{r}') = \frac{1}{4\pi\epsilon_0}\frac{1}{|\vec{r}-\vec{r}'|}$

Solving Gauss' Law equation in differential form to find the Green function

$$\nabla^2 G(\vec{r}, \vec{r}') = -\frac{1}{\epsilon_0}\delta(\vec{r} - \vec{r}')$$