Solving differential equations with Fourier transforms

- Consider a damped simple harmonic oscillator with damping $\gamma$ and natural frequency $\omega_0$ and driving force $f(t)$

$$\frac{d^2y}{dt^2} + 2\gamma \frac{dy}{dt} + \omega_0^2 y = f(t)$$

- At $t = 0$ the system is at equilibrium $y = 0$ and at rest so $\frac{dy}{dt} = 0$
- We subject the system to an force acting at $t = t'$, $f(t) = \delta(t - t')$, with $t' > 0$
- We take $y(t) = \int_{-\infty}^{\infty} g(\omega) e^{i\omega t} d\omega$ and $f(t) = \int_{-\infty}^{\infty} f(\omega) e^{i\omega t} d\omega$
Example continued

• Substitute into the differential equation and we find

\[
[\omega_0^2 - \omega^2 + 2ib\omega] g(\omega) = f(\omega)
\]

• We find also \( f(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(t - t')e^{-i\omega t} dt = \frac{1}{2\pi} e^{-i\omega t'} \)

• We find a relationship between the \( g(\omega) \) and \( f(\omega) \), and then we can write for the response \( g(\omega) \)

\[
g(\omega) = \frac{1}{2\pi} \frac{e^{-i\omega t'}}{\omega_0^2 - \omega^2 + 2ib\omega}
\]

• Then with \( y(t) = 0 \) for \( t < t' \), we get \( y(t) \) for \( t > t' \)

\[
y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega(t-t')}}{\omega_0^2 - \omega^2 + 2ib\omega} d\omega
\]
• The integral is hard to do (we might get to later), but the point is we have reduced the problem to doing an integral
• Assume $b < \omega_0$, then we find for $y(t)$ with $t > t'$,

$$y(t) = e^{-b(t-t')} \frac{\sin [\omega'(t - t')]}{\omega'}$$

where $\omega' = \sqrt{\omega_0^2 - b^2}$ and $y(t) = 0$ for $t < t'$
• You can convince yourself that this is consistent with the $b = 0$ case described in the book (see Eq. 12.5 in chapter 8)
• We can use as an example the damped simple harmonic oscillator subject to a driving force \( f(t) \) (The book example corresponds to \( \gamma = 0 \))

\[
\frac{d^2y}{dt^2} + 2b \frac{dy}{dt} + \omega_0^2 y = f(t)
\]

• Now that we know the properties of the Dirac delta function, we notice that \( f(t) = \int_{-\infty}^{\infty} f(t')\delta(t-t')dt' \)

• This gives a hint that we can treat \( f(t) \) as a sequence of delta-function impulses
Green functions: Damped harmonic oscillator

\[ \frac{d^2 y}{dt^2} + 2b \frac{dy}{dt} + \omega_0^2 y = f(t) \]

• Let’s say \( f(t) \) is zero for \( t < 0 \), and also \( y(t) = 0 \) for \( t < 0 \), and then we turn on the driving force \( f(t) \)

• Using our insight, and the principle of superposition, we assume that the response \( (y(t)) \) depends on the entire history of the force \( f(t') \) from \( 0 < t' < t \),

\[ y(t) = \int_0^t G(t, t')f(t')dt' \]
• Substitute this into the equation of motion

\[ \frac{d^2y}{dt^2} + 2b \frac{dy}{dt} + \omega_0^2 y = f(t) \]

• Use \( y(t) = \int_0^t G(t, t')f(t')dt' \) and \( f(t) = \int_0^\infty f(t')\delta(t' - t)dt' \)

\[ \int_0^t f(t') \left[ \left( \frac{d^2}{dt^2} + 2b \frac{d}{dt} + \omega_0^2 \right) G(t, t') \right] dt' = \int_0^\infty f(t')\delta(t' - t)dt' \]
• We see that the Green function $G(t, t')$ solves the differential equation,

$$\left( \frac{d^2}{dt^2} + 2b \frac{d}{dt} + \omega_0^2 \right) G(t, t') = \delta(t' - t)$$

• Note also that $G(t, t') = 0$ for $t < t'$

• We already solved that! It was just the response $y(t)$ due to a $\delta$-function impulse, with $\omega' = \sqrt{\omega_0^2 - b^2}$

$$G(t, t') = e^{-b(t-t')} \frac{\sin [\omega'(t - t')]}{\omega'}$$

• Notice that the response only depends on $t - t'$, as we expect

• This was for the underdamped case ($b < \omega_0$), and would not work for critical or overdamped cases!
Finally we can write the solution $y(t)$ for any driving force $f(t)$ turned on at $t = 0$, for the damped oscillator in the underdamped regime,

$$y(t) = \int_{0}^{t} G(t, t') f(t') dt' = \int_{0}^{t} e^{-b(t-t')} \frac{\sin [\omega'(t - t')]}{\omega'} f(t') dt'$$
Green functions continued

- Quite powerful! As long as differential equation is linear, we can find the Green (response) function which completely solves any problem
- Another example: Electrostatics
- We know that the electrostatic potential $\phi(\vec{r})$ due to a continuous charge distribution $\rho(\vec{r}')$ is simply additive

$$\phi(\vec{r}) = \frac{1}{4\pi \varepsilon_0} \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3 \vec{r}'$$

- Because of this, Gauss’ Law is a linear differential equation,

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

- Then, since $E = -\vec{\nabla} \phi$, we have

$$\nabla^2 \phi = -\frac{\rho}{\varepsilon_0}$$
Green function for electrostatics

• We will see that \( G(\vec{r}, \vec{r}') = \frac{1}{4\pi\varepsilon_0} \frac{1}{|\vec{r} - \vec{r}'|} \)

• First, take note that \( \rho(\vec{r}) = \int \rho(\vec{r}') \delta(\vec{r} - \vec{r}') d^3\vec{r}' \)

• It might be more clear if we note that \( \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \) and \( \vec{r}' = x'\hat{i} + y'\hat{j} + z'\hat{k} \), and then

\[
\rho(\vec{r}) = \int \int \int \rho(\vec{r}') \delta(x - x') \delta(y - y') \delta(z - z') dx' dy' dz'
\]

• Next we use that the potential \( \phi(\vec{r}) \) is found just by adding up the contributions due to each part of \( \rho(\vec{r}') \), so

\[
\phi(\vec{r}) = \int G(\vec{r}, \vec{r}') \rho(\vec{r}') d^3\vec{r}'
\]
• Substitute into the Gauss Law expression $\nabla^2 \phi = -\frac{\rho}{\epsilon_0}$

$$\int \nabla^2 G(\vec{r}, \vec{r}') \rho(\vec{r}') d^3\vec{r}' = -\frac{1}{\epsilon_0} \int \rho(\vec{r}') \delta(\vec{r} - \vec{r}') d^3\vec{r}'$$

• Noting that the $\nabla^2$ is with respect to $\vec{r}$ (and not $\vec{r}'$, we get the equation for the Green function

$$\nabla^2 G(\vec{r}, \vec{r}') = -\frac{1}{\epsilon_0} \delta(\vec{r} - \vec{r}')$$

• Then $G(\vec{r}, \vec{r}')$ is just the potential at $\vec{r}$ due to a unit charge located at $\vec{r}'$

• Since we know Coulomb’s Law, we can see right away that

$$G(\vec{r}, \vec{r}') = \frac{1}{4\pi\epsilon_0} \frac{1}{|\vec{r} - \vec{r}'|}$$
Solving Gauss’ Law equation in differential form to find the Green function

\[ \nabla^2 G(\vec{r}, \vec{r}') = -\frac{1}{\epsilon_0} \delta(\vec{r} - \vec{r}') \]