Even and odd functions

- We define an even function such that $f(-x) = f(x)$
- We define an odd function such that $f(-x) = -f(x)$
- Example, $\sin x$ is an odd function because $\sin -x = -\sin x$
- Example, $\cos x$ is an even function because $\cos -x = \cos x$
- Now consider a Fourier series of a periodic, even function $f(x)$ ($f(-x) = f(x)$), over the interval $-\pi < x < \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

- Now consider the integrals to determine the coefficients, first a_0

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)dx$$

Even/odd functions continued

- Next the a_n for finite n, we again use the fact that $f(x)$ is even, and also $\cos nx$ is even,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx dx$$

- Next we can show $b_n = 0$ when $f(x)$ is even,

$$b_n = \frac{1}{\pi}\left[\int_{0}^{\pi} f(x)\sin nx dx - \int_{0}^{\pi} f(x)\sin nx dx\right] = 0$$

Even/odd functions continued

- We can also treat the odd case $f(-x) = -f(x)$, then

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = 0$$

$$a_n = \frac{1}{\pi} \left[\int_0^{\pi} f(x) \cos nx dx - \int_0^{\pi} f(x) \cos nx dx \right] = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx$$

Example of an odd function

- The step function provides us an example of an odd function
- For the step function, we found $a_0 = 1$, so actually it is neither odd nor even, but if we define the step function as $f(x) = -1/2$ for $-\pi < x < 0$ and $f(x) = 1/2$ for $0 < x < \pi$, then $f(x)$ is odd and

$$f(x) = \frac{2}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + ...\right)$$

- The point is that the $\cos nx$ terms vanish ($a_n = 0$ for all n)
- The advantage is that we could have *predicted* that $a_n = 0$ for all n even before trying to do the integral

Another example: even function

- Find the expansion for $f(x) = x^2$ on the interval $-\pi < x < \pi$, periodically repeating with period 2π
- For this case, $f(x)$ is clearly even since $f(-x) = (-x)^2 = x^2 = f(x)$, hence $b_n = 0$ for each n, and

$$a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx dx$$

- We find $a_0 = \frac{2\pi^2}{3}$
- For the other n, we evaluate the integral (homework!)

$$\frac{2}{\pi} \int_0^{\pi} x^2 \cos nx dx = \frac{4}{n^2}(-1)^n$$

$$f(x) = \frac{\pi^2}{3} + 4\left[-\frac{\cos x}{1} + \frac{\cos 2x}{4} - \frac{\cos 3x}{9} + \ldots\right]$$

Parseval's theorem

- For a periodic function $f(x)$ defined on $-l < x < l$, we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

- The average of $[f(x)]^2$ is $\frac{1}{2l}\int_{-l}^{l}[f(x)]^2 dx$
- To obtain Parseval's theorm, use the integrals we obtained before

$$\frac{1}{2l}\int_{-l}^{l} \sin\frac{m\pi x}{l} \sin\frac{n\pi x}{l} dx = \frac{1}{2}\delta_{m,n}$$

$$\frac{1}{2l}\int_{-l}^{l} \cos\frac{m\pi x}{l} \cos\frac{n\pi x}{l} dx = \frac{1}{2}\delta_{m,n}$$

$$\frac{1}{2l}\int_{-l}^{l} \sin\frac{m\pi x}{l} \cos\frac{n\pi x}{l} dx = 0$$

Parseval's theorem continued

- Using the previous integrals, we find

$$\frac{1}{2l}\int_{-l}^{l}[f(x)]^2 dx = \left(\frac{1}{2}a_0\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

- Example: Problem 5.8 and Problem 11.7
- Find the Fourier series for $f(x) = 1 + x$ defined on $-\pi < x < \pi$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(1+x)dx = 2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(1+x)\cos nx dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(1+x)\sin nx dx = \frac{2(-1)^{n+1}}{n}$$

Example of Parseval's theorem continued

- Then Parseval's theorem states,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(1+x)^2dx = 1 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{4}{n^2} = 1 + 2\sum_{n=1}^{\infty}\frac{1}{n^2}$$

- Problem 11.8 asks us to evaluate $\sum_{n=1}^{\infty}\frac{1}{n^2}$, and from Parseval's theorem we see that,

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = -\frac{1}{2} + \frac{1}{4\pi}\int_{-\pi}^{\pi}(1+x)^2dx = \frac{\pi^2}{6}$$

- Might even use to compute π!

$$\pi = \sqrt{6}\left[\sum_{n=1}^{\infty}\frac{1}{n^2}\right]^{1/2}$$

For the fun of it... π

• Exact value of $\pi = 3.141592653589793$ (Correct to 16 digits... my computer using intrinsic functions got the digits after these incorrect)

• From serious on previous page, I got the following results:

10^4 terms: 3.141497163947214

10^5 terms: 3.141583104326456

10^6 terms: 3.141591698660508

10^7 terms: 3.141592558095902

• Correct to 7 digits for 10^7 terms, and took < 1 second to compute

Parseval's theorem for complex Fourier series

- When we average $|f(x)|^2 = f^*(x)f(x)$ over one period, we obtain $\sum_{n=-\infty}^{\infty} |c_n|^2$
- Proof in problem 3, for $f(x)$ periodic with periodicity 2π $(-\pi < x < \pi)$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

- We use the orthogonality of the functions e^{inx},

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(n-m)x} dx = \delta_{m,n}$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f^*(x) f(x) dx = \frac{1}{2\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_m^* c_n \int_{-\pi}^{\pi} e^{i(m-n)x} dx = \sum_{n=-\infty}^{\infty} c_n^* c_n$$

Another example: problem 2

• We can also average $[f(x)]^2$ using the complex series (contrast to averaging $|f(x)|^2 = f^*(x)f(x)$)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2 dx = \frac{1}{2\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_m c_n \int_{-\pi}^{\pi} e^{i(m+n)x} dx = \sum_{n=-\infty}^{\infty} c_n c_{-n}$$

• Consider the special case where $f(x)$ is real, then the expansion in complex Fourier series is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} = c_0 + \sum_{n=1}^{\infty}\left(c_n e^{inx} + c_{-n} e^{-inx}\right)$$

• Since $f(x)$ is real, the complex parts must cancel, so using the Euler formula

Problem 2 continued

$$f(x) = c_0 + \sum_{n=1}^{\infty} (c_n + c_{-n}) \cos nx + \sum_{n=1}^{\infty} (ic_n - ic_{-n}) \sin nx$$

- For the imaginary parts to go away, we require $c_{-n} = c_n^*$

$$c_n + c_{-n} = c_n + c_n^* = 2Re[c_n]$$

$$ic_n - ic_{-n} = ic_n - ic_n^* = -2Im[c_n]$$

- Then for real $f(x)$, we obtain

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} [f(x)]^2 dx = \frac{1}{2\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_m c_n \int_{-\pi}^{\pi} e^{i(m+n)x} dx = \sum_{n=-\infty}^{\infty} c_n^* c_n$$