Chapter 6: Vector Analysis

We use derivatives and various products of vectors in all areas of physics. For example, Newton's 2nd law is $\vec{F} = m\frac{d^2\vec{r}}{dt^2}$. In electricity and magnetism, we need surface and volume integrals of various fields. Fields can be scalar in some cases, but often they are vector fields like $\vec{E}(x, y, z)$ and $\vec{B}(x, y, z)$
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- Divergence theorem, Green theorem in plane, and Stokes theorem

Application of vector multiplication

• We have seen the dot or scalar product, which in a Cartesian system we can write as

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

• We also saw that $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta$
• An example is computing work W (a scalar) due to a force $\vec{F}$ along a path $d\vec{r}$

$$dW = \vec{F} \cdot d\vec{r}$$

Vector or cross products

- We saw for cross products, or vector products

$$\vec{A} \times \vec{B} = \vec{C}$$

- We saw $\vec{C}$ is perpendicular to $\vec{A}$ and $\vec{B}$, and $|\vec{C}| = |\vec{A}||\vec{B}| \sin\theta$

Application for cross or vector products

• In computing torque $\vec{\tau}$ about some point O, we use a cross product

$$\vec{\tau} = \vec{r} \times \vec{F}$$

• From this relationship we can find the change in angular momentum with time $\tau = \frac{d\vec{L}}{dt}$ (For example, $\vec{L} = I\vec{\omega}$)
• We might have the axis of rotation *not* parallel to the torque $\tau = \vec{r} \times \vec{F}$
• In this case, if $\hat{n}$ is the direction of the axis (unit vector), then the torque about that axis is (with the point O somewhere along the axis)

$$\tau = \hat{n} \cdot (\vec{r} \times \vec{F})$$

Example: Torque about a point and a line, problem 19

• For a force $\vec{F} = \hat{i} + 3\hat{j} + 2\hat{k}$ acting at the point (1,1,1), find the torque about the point (2,-1,5)

• We find the vector $\vec{r}$ from

$$\vec{r} = (1-2)\hat{i} + (1+1)\hat{j} + (1-5)\hat{k} = -\hat{i} + 2\hat{j} - 4\hat{k}$$

• Then we find the torque from $\vec{r} \times \vec{F}$

$$\tau = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & -4 \\ 1 & 3 & 2 \end{vmatrix} = 16\hat{i} - 2\hat{j} - 5\hat{k}$$

Torque about about a line, continuation of problem 19

- In problem 19b, we want torque about the line

$$\vec{n} = 2\hat{i} - \hat{j} + 5\hat{k} + (\hat{i} - \hat{j} + 2\hat{k})t$$

- This is a parametric equation with t a continuous parameter
- The unit vector $\hat{n} = \frac{1}{\sqrt{6}}\hat{i} - \frac{1}{\sqrt{6}}\hat{j} + \frac{2}{\sqrt{6}}\hat{k}$
- From $t = 0$ notice the line passes through the point (2,-1,5)
- We can still find the vector $\vec{r}$ to the point on the line (2,-1,5), so $\vec{r} = -\hat{i} + 2\hat{j} - 4\hat{k}$

$$\tau = \hat{n} \cdot (\vec{r} \times \vec{F}) = (\frac{1}{\sqrt{6}}\hat{i} - \frac{1}{\sqrt{6}}\hat{j} + \frac{2}{\sqrt{6}}\hat{k}) \cdot (16\hat{i} - 2\hat{j} - 5\hat{k}) = \frac{8}{\sqrt{6}}$$

- Notice we get a scalar, which is the component of τ along the $\hat{n}$ direction. We could also write $\vec{\tau} = \tau\hat{n} = \frac{4}{3}\hat{i} - \frac{4}{3}\hat{j} + \frac{8}{3}\hat{k}$

Relationship between linear and angular momentum and velocity

- The relationship between linear and angular velocity of a point in a rotating rigid body is given by,

$$\vec{v} = \vec{\omega} \times \vec{r}$$

- The relationship between angular momentum and linear momentum is given by

$$\vec{L} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v}$$

- We can combine these relations to get,

$$\vec{L} = m\vec{r} \times (\vec{\omega} \times \vec{r})$$

Centripetal acceleration

- For a mass rotating with constant angular velocity $\vec{\omega}$, the centripetal acceleration is

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{\omega} \times \vec{r}) = \vec{\omega} \times \frac{d\vec{r}}{dt} = \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

Example of centripetal acceleration, Section 3, Problem 17

- If $\vec{r}$ is perpendicular to $\vec{\omega}$, show $\vec{a} = -\omega^2 \vec{r}$ and show the acceleration is toward the axis of rotation with magnitude $a = \frac{v^2}{r}$
- Start with $\vec{a} = \vec{\omega} \times (\vec{\omega} \times \vec{r})$
- Since $\vec{\omega}$ and $\vec{r}$ are perpendicular, $\theta = \pi/2$
- Then $|\vec{\omega} \times \vec{r}| = \omega r$
- $\vec{\omega} \times \vec{r}$ is also perpendicular to $\vec{\omega}$, so $|\vec{\omega} \times (\vec{\omega} \times \vec{r})| = \omega^2 r$
- From the right-hand rule it is easy to see then that $\vec{a} = -\omega^2 \vec{r}$
- Finally $v = |\vec{v}| = |\vec{\omega} \times \vec{r}| = \omega r$, or $\omega = v/r$ and $a = \omega^2 r = \frac{v^2}{r}$

Triple products

• We have already seen that the volume of a parallelpiped from $\vec{A}$, $\vec{B}$, and $\vec{C}$ can be found

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

• It is also useful to be able to find the vector product $\vec{A} \times (\vec{B} \times \vec{C})$

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$$

Differentiation of vectors

• In a Cartesian system, $\hat{i}$, $\hat{j}$, and $\hat{k}$ are fixed unit vectors
• If we have a vector $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$, where the components are functions of t, then we can take a derivative

$$\frac{d\vec{A}}{dt} = \frac{dA_x}{dt}\hat{i} + \frac{dA_y}{dt}\hat{j} + \frac{dA_z}{dt}\hat{k}$$

• For example, $\vec{A}$ could be the position of a particle, and then the time derivative is the velocity. If $\vec{A}$ is the velocity, then the time derivative is the acceleration.
• What do we do if the vector is described in another coordinate systems that does not have fixed unit vectors?

Differentiation of vectors in a polar system

• We can express $\vec{A}$ in the xy-plane using unit vectors in a polar system

$$\vec{A} = A_x\hat{i} + A_y\hat{j} = A_r\hat{e}_r + A_\theta\hat{e}_\theta$$

• The $\hat{i}$ and $\hat{j}$ unit vectors have a fixed direction, but $\hat{e}_r$ and $\hat{e}_\theta$ do not