Change of variables in the integral; Jacobian

• Element of area in Cartesian system, $dA = dxdy$

• We can see in polar coordinates, with $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$, and $\tan\theta = y/x$, that $dA = rdrd\theta$

• In three dimensions, we have a volume $dV = dxdydz$ in a Carestian system

• In a cylindrical system, we get $dV = rdrd\theta dz$

• In a spherical system, we get $dV = r^2 drd\phi d(\cos\theta)$

• We can find with simple geometry, but how can we make it systematic?

• We can define the Jacobian to make this more straightforward and automatic

The Jacobian

- In a Cartesian system we find a volume element simply from $dV = dxdydz$
- Now assume $x \rightarrow x(u,v,w)$, $y \rightarrow y(u,v,w)$, and $z \rightarrow z(u,v,w)$
- We have in the Cartesian system $d\vec{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$
- We can then find the total differentials dx, dy, and dz from

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv + \frac{\partial x}{\partial w}dw$$

$$dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv + \frac{\partial y}{\partial w}dw$$

$$dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv + \frac{\partial z}{\partial w}dw$$

Jacobian continued

• We can define $\vec{A}$ to be along a direction such that $dv = dw = 0$, then in the Cartesian system

$$\vec{A} = \left(\hat{i}\frac{\partial x}{\partial u} + \hat{j}\frac{\partial y}{\partial u} + \hat{k}\frac{\partial z}{\partial u} \right) du$$

• Likewise $\vec{B}$ will be along a direction with $du = dw = 0$, then in the Cartesian system we see,

$$\vec{B} = \left(\hat{i}\frac{\partial x}{\partial v} + \hat{j}\frac{\partial y}{\partial v} + \hat{k}\frac{\partial z}{\partial v} \right) dv$$

• Finally $\vec{C}$ will be along a direction where $du = dv = 0$, then in the Cartesian system we see,

$$\vec{C} = \left(\hat{i}\frac{\partial x}{\partial w} + \hat{j}\frac{\partial y}{\partial w} + \hat{k}\frac{\partial z}{\partial w} \right) dw$$

Jacobian continued

• The volume element made by these vectors is $dV = \vec{A} \cdot (\vec{B} \times \vec{C})$, which is simply the determinant

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix} dudvdw = Jdudvdw$$

• Here the determinant is the Jacobian J

• We have to be careful! The J found above might be negative, so in general we take $|J|$

• Notice also that we can interchange rows and columns (i.e. take the transpose) and the determinant is unchanged, so

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Example: Volume element in cylindrical coordinates

• We know that $dV = dxdydz$ in Cartesian coordinates, and also $dV = rdrd\cos\theta dz$ in cylindrical coordinates, but let's prove it!

• We see that $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$

• We then can find J,

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

• So finally the element of volume $dV = Jdrd\theta dz = rdrd\theta dz$ in cylindrical coordinates

• The book proves that $dV = r^2 drd\phi d(\cos\theta)$ in Section 4, go through the proof to practice Jacobians!

Element of area

- We might have an integral over area $dA = dxdy$, and want instead the integral in some other coordinate system
- Again assume we have $x \rightarrow x(u,v)$ and $y \rightarrow y(u,v)$
- Define vectors $\vec{B}$ and $\vec{C}$ which will lie in the x,y plane
- For $\vec{B}$ we assume v does not change

$$\vec{B} = \left(\hat{i}\frac{\partial x}{\partial u} + \hat{j}\frac{\partial y}{\partial u}\right) du$$

- For $\vec{C}$ we assume u does not change

$$\vec{C} = \left(\hat{i}\frac{\partial x}{\partial v} + \hat{j}\frac{\partial y}{\partial v}\right) dv$$

- An element of area is found from $dA = |\vec{B} \times \vec{C}|$

Element of area continued

- We find for $\vec{B} \times \vec{C}$

$$\vec{B}\times\vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} dudv = \hat{k} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} dudv = \hat{k} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} dudv$$

- We define the Jacobian J as

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

- Again accounting for the fact that J may be negative, we find for dA

$$dA = |J| dudv$$

Example: Surface integral in polar coordinates

• We know that $dA = dxdy$, and in polar coordinates $dA = rdrd\theta$, but let's use the Jacobian to define

• We have $x = r\cos\theta$ and $y = r\sin\theta$, so we have for J

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

• So we find as we expected for dA

$$dA = |J| drd\theta = rdrd\theta$$

Elements of length

- We might need elements of arc lengths in line integrals
- In Cartesian coordinates, it is quite straightforward

$$ds^2 = dx^2 + dy^2 + dz^2$$

• To find in another system, we need dx in terms of the other system, so $x \to x(u, v, w)$, etc.

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv + \frac{\partial x}{\partial w} dw$$

Example in cylindrical coordinates

- For example, in cylindrical coordinates, we have $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$, so

$$dx = \cos\theta dr - r\sin\theta d\theta$$

$$dy = \sin\theta dr + r\cos\theta d\theta$$

$$dz = dz$$

- So we find the element of arc length in cylindrical coordinates,

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2$$

Example in spherical coordinates

- In spherical coordinates we have $x = r\cos\phi\sin\theta$,
$y = r\sin\phi\sin\theta$, and $z = r\cos\theta$
- An element of arc length becomes,

$$ds^2 = dr^2 + r^2 d\theta + r^2 \sin^2\theta d\phi^2$$

Surface integrals on a cylinder or a sphere

- We can see that an element $\vec{dA}$ with a magnitude equal to the area and direction normal to the surface can be found in a cylindrical system by noticing that the $\hat{z}dz$ and $\hat{\theta}ad\theta$ vectors are perpendicular, so

$$\vec{dA} = \hat{\theta}ad\theta \times \hat{z}dz = ad\theta dz\hat{r}$$

- Obviously the magnitude is $dA = ad\theta dz$
- Likewise in spherical coordinates we find $\vec{dA}$ from

$$\vec{dA} = a\hat{\phi}\sin\theta d\phi \times a\hat{\theta}d\theta = a^2\sin\theta d\phi d\theta\hat{r}$$

- In spherical coordinates the magnitude is $dA = a^2\sin\theta d\phi d\theta$

Example: Center of mass

• We can find the center of mass coordinates $\bar{x}$, $\bar{y}$, and $\bar{z}$ defined by, in the case of a continuous mass distribution

$$\bar{x} = \frac{\int x dM}{\int dM}$$

$$\bar{y} = \frac{\int y dM}{\int dM}$$

$$\bar{z} = \frac{\int z dM}{\int dM}$$

• The significance is that when no external forces are acting on the body, the center of mass moves with a uniform velocity (or is at rest)

More significance of the center of mass

- If there is a total (net) force $\vec{F}_{net}$, then we have

$$M\frac{d^2\bar{x}}{dt^2} = F_{net,x}$$

$$M\frac{d^2\bar{y}}{dt^2} = F_{net,y}$$

$$M\frac{d^2\bar{z}}{dt^2} = F_{net,z}$$

Example with constant density

• With a constant density, the center of mass corresponds to the centroid of the body

• Section 3, problem 7, Find the center of mass $\bar{x}$ and $\bar{y}$ for a rectangular lamina with constant areal density $\rho = 1$ and vertices at $(0,0)$, $(0,2)$, $(3,0)$, and $(3,2)$

• The factor $dM = \rho dxdy = dxdy$ (since $\rho = 1$)

• The limits on x integration are 0 and 3, and the limits on y integration are 0 and 2, so

$$\bar{x} = \frac{\int_0^2 \int_0^3 x dxdy}{\int_0^2 \int_0^3 dxdy} = \frac{9}{6} = \frac{3}{2}$$

$$\bar{y} = \frac{\int_0^2 \int_0^3 y dxdy}{\int_0^2 \int_0^3 dxdy} = \frac{6}{6} = 1$$

• Not surprising, the center of mass is the centroid and is right in the middle of rectangle

Example continued

- What if $\rho = xy$? (This is the case in problem 7)

$$\bar{x} = \frac{\int_0^2 \int_0^3 x^2 y dx dy}{\int_0^2 \int_0^3 xy dx dy} = 2$$

$$\bar{y} = \frac{\int_0^2 \int_0^3 x y^2 dx dy}{\int_0^2 \int_0^3 xy dx dy} = \frac{4}{3}$$

Moment of inertia of a solid cylinder

• Consider a cylinder of height h, radius R, and mass M. Mass density is uniform.
• The volume of the cylinder is $V = \pi R^2 h$, so
$\rho = M/V = M/(\pi R^2 h)$
• Use cylindrical coordinates and determine the moment of inertia about the z axis I_z

$$I_z = \rho \int_0^h \int_0^{2\pi} \int_0^R r^3 dr d\theta dz = \frac{M}{\pi R^2 h} \frac{2\pi R^4 h}{4} = MR^2$$

We use derivatives and various products of vectors in all areas of physics. For example, Newton's 2nd law is $\vec{F} = m\frac{d^2\vec{r}}{dt^2}$. In electricity and magnetism, we need surface and volume integrals of various fields. Fields can be scalar in some cases, but often they are vector fields like $\vec{E}(x, y, z)$ and $\vec{B}(x, y, z)$
By the end of the chapter you should be able to

- Work with various vector products including triple products

We use derivatives and various products of vectors in all areas of physics. For example, Newton's 2nd law is $\vec{F} = m\frac{d^2\vec{r}}{dt^2}$. In electricity and magnetism, we need surface and volume integrals of various fields. Fields can be scalar in some cases, but often they are vector fields like $\vec{E}(x, y, z)$ and $\vec{B}(x, y, z)$

By the end of the chapter you should be able to

- Work with various vector products including triple products
- Differentiate vectors

We use derivatives and various products of vectors in all areas of physics. For example, Newton's 2nd law is $\vec{F} = m\frac{d^2\vec{r}}{dt^2}$. In electricity and magnetism, we need surface and volume integrals of various fields. Fields can be scalar in some cases, but often they are vector fields like $\vec{E}(x,y,z)$ and $\vec{B}(x,y,z)$
By the end of the chapter you should be able to

- Work with various vector products including triple products
- Differentiate vectors
- Use directional derivatives and the gradient

Chapter 6: Vector Analysis

We use derivatives and various products of vectors in all areas of physics. For example, Newton's 2nd law is $\vec{F} = m\frac{d^2\vec{r}}{dt^2}$. In electricity and magnetism, we need surface and volume integrals of various fields. Fields can be scalar in some cases, but often they are vector fields like $\vec{E}(x, y, z)$ and $\vec{B}(x, y, z)$

By the end of the chapter you should be able to

- Work with various vector products including triple products
- Differentiate vectors
- Use directional derivatives and the gradient
- Divergence and curl

We use derivatives and various products of vectors in all areas of physics. For example, Newton's 2nd law is $\vec{F} = m\frac{d^2\vec{r}}{dt^2}$. In electricity and magnetism, we need surface and volume integrals of various fields. Fields can be scalar in some cases, but often they are vector fields like $\vec{E}(x, y, z)$ and $\vec{B}(x, y, z)$
By the end of the chapter you should be able to

- Work with various vector products including triple products
- Differentiate vectors
- Use directional derivatives and the gradient
- Divergence and curl
- Line integrals

Chapter 6: Vector Analysis

We use derivatives and various products of vectors in all areas of physics. For example, Newton's 2nd law is $\vec{F} = m\frac{d^2\vec{r}}{dt^2}$. In electricity and magnetism, we need surface and volume integrals of various fields. Fields can be scalar in some cases, but often they are vector fields like $\vec{E}(x, y, z)$ and $\vec{B}(x, y, z)$
By the end of the chapter you should be able to

- Work with various vector products including triple products
- Differentiate vectors
- Use directional derivatives and the gradient
- Divergence and curl
- Line integrals
- Divergence theorem, Green theorem in plane, and Stokes theorem

Triple products

• We have already seen that the volume of a parallelpiped from $\vec{A}$, $\vec{B}$, and $\vec{C}$ can be found

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & 0 \end{vmatrix}$$

• It is also useful to be able to find the vector product $\vec{A} \times (\vec{B} \times \vec{C})$

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$$