Total differential for \( y = f(x) \)

- For \( y = f(x) \), we have \( y' = \frac{dy}{dx} = \frac{df}{dx} \).
- We can treat \( dx = \Delta x \) as an independent variable.
- In the limit \( \Delta x \to 0 \), then
  \[
  \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}
  \]
- If \( \Delta x \) finite, then \( dy \) is not exactly \( \Delta y \).
Total differential for \( z = f(x, y) \) and for many independent variables

- For a function of two variables, \( z = f(x, y) \), we can define the total differential

\[
dz = \frac{\partial z}{\partial x} dx + \frac{\partial}{\partial y} dy
\]

- We can have \( dx \) and \( dy \) independent variables
- Then \( dz \) is the change in \( z \) along the tangent plane at \( x, y \)
- As with the previous example, \( dz \) is not equal to \( \Delta z \) for finite \( dx \) and \( dy \)
- For a function of many variables \( u = f(x_1, x_2, \ldots, x_N) \), we define the total differential

\[
du = \sum_{n=1}^{N} \frac{\partial u}{\partial x_n} dx_n
\]
In thermodynamics, we have quantities that might pressure $p$, volume $V$, temperature $T$, entropy $S$, particle number $N$, and chemical potential $\mu$.

These are not all independent, so if we know $p$ then $V$ is determined, hence we describe quantities in terms of some subset of all the possible variables (In fact, $p$ and $V$ are conjugate pairs, as are $T$ and $S$, and also $N$ and $\mu$.)

The total energy $U(S, V, N)$, so

$$dU = \frac{\partial U}{\partial S} dS + \frac{\partial U}{\partial V} dV + \frac{\partial U}{\partial N} dN$$

We define $T = \frac{\partial U}{\partial S}$, $p = -\frac{\partial U}{\partial V}$, and $\mu = \frac{\partial U}{\partial N}$

$$dU = TdS - pdV + \mu dN$$
Legendre transformations

- Construct a new function $F = U - TS$, then

$$dF = dU - TdS - SdT = -SdT - pdV + \mu dN$$

- We see that $F(T, V, N)$, different independent variables!
- This is an example of a Legendre transformation
- Consider another example, $G = F + pV$, so

$$dG = dF + pdV + Vdp = -SdT + Vdp + \mu dN$$

- The thermodynamics function $G(T, p, N)$ is quite convenient because experiments are done usually at constant $T$ and $p$
• Because the total differential $du$ is close to $\Delta u$ for small changes in the independent variables, we can use for approximation

• Example: Find the approximation value of $\frac{1}{\sqrt{0.25-10^{-20}}} - \frac{1}{\sqrt{0.25}}$

Here $y = f(x) = \frac{1}{\sqrt{x}}$, so we want $dy = \frac{df}{dx} dx = -\frac{1}{2}x^{-3/2} dx$, so

$$\frac{1}{\sqrt{0.25 + 10^{-20}}} - \frac{1}{\sqrt{0.25}} \approx -\frac{1}{2}(0.25)^{-3/2}(-10^{-20}) = 4 \times 10^{-20}$$
While $dy \neq \Delta y$ for $dx = \Delta x$ and $y = f(x)$, $dy$ may still be a good approximation for $\Delta y$. As an example:

- For very large $n$, show that $\frac{1}{(n+1)^3} - \frac{1}{n^3} \simeq -\frac{3}{n^4}$

In this case, we consider the function $y = f(x) = \frac{1}{x^3}$. We get $y' = -\frac{3}{x^4}$. From above, it is obvious that we should consider $x = n$ and $dx = 1$. So,

$$dy = y' dx = -\frac{3}{n^4} \simeq \frac{1}{(n + 1)^3} - \frac{1}{n^3}$$

Why does $n$ have to be big for this to work? Consider a Taylor expansion of $f(x)$ at $x = n$ and then the lowest order correction to our approximation, which scales as $n^{-5}$. For $n = 2$, lowest order correction is equal to our approximation! So $n \gg 2$ required.
Another example....

- Using differentials, *estimate* the energy needed to move a 10 kg mass from a point with coordinates \((4000\, km, 4000\, km, 3000\, km)\) to a point with coordinates given by \((4020\, km, 4050\, km, 3010\, km)\). The center of the Earth corresponds to \((0, 0, 0)\).
- The gravitational potential (energy per unit mass) \(\Phi(r)\) is

\[
\Phi(r) = -\frac{GM}{r}
\]

where \(r = (x^2 + y^2 + z^2)^{\frac{1}{2}}\), and \(G = 6.67 \times 10^{-11}\ \text{Nm}^2/\text{kg}^2\), and the mass of the Earth is \(M = 5.97 \times 10^{24} \text{ kg}\).

\[
d\Phi = \frac{\partial \Phi}{\partial x} \, dx + \frac{\partial \Phi}{\partial y} \, dy + \frac{\partial \Phi}{\partial z} \, dz
\]
\[ d\Phi = -\frac{GM}{r^3} (xdx + ydy + zdz) \]

- Then \( dW = -md\Phi \) where \( m = 10\text{kg} \)
- To do the calculation, \( x = 4 \times 10^6\text{m} \), \( y = 4 \times 10^6\text{m} \), \( z = 3 \times 10^6\text{m} \), \( dx = 2 \times 10^4\text{m} \), \( dy = 5 \times 10^4\text{m} \), and \( dz = 1 \times 10^4\text{m} \)
- Exact answer is just \( dW = \frac{GmM}{r_1} - \frac{GmM}{r_2} \), where \( r_1 \) (\( r_2 \)) is the starting (final) distance from the center of the Earth
We can think in terms of differentials, for example.

Example: Find $\frac{dz}{dt}$ for $z = (\sin t)^{\tan^{-1} t}$

Take $x = \sin t$ and $y = \tan^{-1} t$, then $z = x^y$ and,

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = yx^{y-1} dx + x^y \ln x dy$$

Then, since $x = \sin t$, we have $dx = \cos t dt$, and $y = \tan^{-1} t$ so $dy = \frac{dt}{1+t^2}$, so finally

$$\frac{dz}{dt} = yx^{y-1} \cos t + x^y \ln x \frac{1}{1 + t^2}$$
Partial differentiation for max/min problems

• For \( y = f(x) \), we can find maxima and minima from solving \( \frac{df}{dx} = 0 \)
• Example: Find the maxima/minima of \( y = f(x) = x^2 - 2x + 3 \)

\[
\frac{df}{dx} = 2x - 2 = 0
\]

This is solved by \( x = 1 \). But is it a maxima or a minima? We find \( \frac{d^2f}{dx^2} = 2 \) is positive, so this is a minima.
• For \( z = f(x, y) \), might have a maxima, minima, or a saddle point. For example, \( \frac{d^2}{dx^2} > 0 \) and \( \frac{d^2}{dy^2} < 0 \) would represent a saddle point.
Example of max/min problem in two dimensions

• Section 8, problem 3. Find the extrema of

\[ f(x, y) = x^2 + y^2 + 2x - 4y + 10 \]

\[
\frac{\partial f}{\partial x} = 2x + 2 = 0
\]

\[
\frac{\partial f}{\partial y} = 2y - 4 = 0
\]

• There is a minima at \( x = -1, y = 2 \). It is a minima since

\[
\frac{\partial^2 f}{\partial x^2} > 0 \text{ and } \frac{\partial^2 f}{\partial y^2} > 0
\]