The Poisson equation can be written,

$$\nabla^2 u(\vec{r}) = \rho(\vec{r})$$

For example, in two dimensions, $u(x, y)$ and $\rho(x, y)$

$$\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] u(x, y) = \rho(x, y)$$

For the case of electrostatics, using Gaussian units we have

$$\nabla^2 \phi(\vec{r}) = -4\pi \rho(\vec{r})$$

For a point charge, $\nabla^2 \phi(\vec{r}) = -4\pi \delta(\vec{r} - \vec{r}')$, and we find

$$\phi(\vec{r}) = \frac{1}{|\vec{r} - \vec{r}'|}$$

The function $G(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|}$ is the Green function for the Poisson equation
The Green function \( G(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|} \) solves the Poisson equation for a point source,

\[
\nabla^2 G(\vec{r}, \vec{r}') = -4\pi \delta(\vec{r} - \vec{r}')
\]

Now we notice that \( \rho(\vec{r}) = \int \int \int \rho(\vec{r}') \delta(\vec{r} - \vec{r}') \, d\tau \), and then with the volume integral extending over the region of nonzero charge density,

\[
\phi(\vec{r}) = \int \int \int G(\vec{r}, \vec{r}') \rho(\vec{r}') \, d\tau
\]

Explicitly, this is just

\[
\phi(\vec{r}) = \int \int \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} \, d\tau
\]
Note that the Green function above gives zero at infinity, but we may not always want that boundary condition.

We can always add any function $F(\vec{r}, \vec{r}')$ that satisfies Laplace’s equation,

\[ \nabla^2 F(\vec{r}, \vec{r}') = 0 \]

So more generally, $G(\vec{r}, \vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} + F(\vec{r}, \vec{r}')$.