Poisson equation

- The Poisson equation can be written,

$$\nabla^2 u(\vec{r}) = \rho(\vec{r})$$

- For example, in two dimensions, $u(x, y)$ and $\rho(x, y)$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] u(x, y) = \rho(x, y)$$

- For the case of electrostatics, using Gaussian units we have

$$\nabla^2 \phi(\vec{r}) = -4\pi\rho(\vec{r})$$

- For a point charge, $\nabla^2 \phi(\vec{r}) = -4\pi\delta(\vec{r} - \vec{r}')$, and we find $\phi(\vec{r}) = \frac{1}{|\vec{r}-\vec{r}'|}$
- The function $G(\vec{r}, \vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|}$ is the Green function for the Poisson equation

Green function for the Poisson equation

• The Green function $G(\vec{r},\vec{r}')=\frac{1}{|\vec{r}-\vec{r}'|}$ solves the Poisson equation for a point source,

$$\nabla^2 G(\vec{r},\vec{r}') = -4\pi\delta(\vec{r}-\vec{r}')$$

• Now we notice that $\rho(\vec{r}) = \int\int\int \rho(\vec{r}')\delta(\vec{r}-\vec{r}')d\tau$, and then with the volume integral extending over the region of nonzero charge density,

$$\phi(\vec{r}) = \int\int\int G(\vec{r},\vec{r}')\rho(\vec{r}')d\tau$$

• Explicitly, this is just

$$\phi(\vec{r}) = \int\int\int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}d\tau$$

Green function for the Poisson equation, continue

• Note that the Green function above gives zero at infinity, but we may not always want that boundary condition

• We can always add any function $F(\vec{r}, \vec{r}')$ that satisfies Laplace's equation,

$$\nabla^2 F(\vec{r}, \vec{r}') = 0$$

• So more generally, $G(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|} + F(\vec{r}, \vec{r}')$