Green function for diffusion equation

- Consider the diffusion equation,

\[
\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2} \frac{\partial u}{\partial t}
\]

- Let’s solve using the inverse Fourier transform,

\[
u(x, t) = \int_{-\infty}^{\infty} u(k, t)e^{ikx} \, dx
\]

- Substitution in the differential equation allows us to easily integrate the time dependence...

\[
-k^2 u(k, t) = \frac{1}{\alpha^2} \frac{\partial u(k, t)}{\partial t}
\]

- We obtain by integration \( u(k, t) = u(k, t = t')e^{-\alpha^2 k^2(t-t')} \)
- We could also do it by separation of variables, practice it that way too....
Green function for diffusion equation, continued

- Assume we have a point source at \( t = t' \), so that
  \[ u(x, t = t') = \delta(x - x') \]
- We can then find \( u(k, t = t') \) for the Fourier transform of the point source
  \[
  u(k, t = t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(x - x') e^{-ikx} \, dx = \frac{e^{-ikx'}}{2\pi}
  \]
- Finally we find \( u(x, t) \) for \( t > t' \) from the inverse Fourier transform
  \[
  u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} e^{-k^2\alpha^2(t-t')} \, dk
  \]
- The integral can be done by “completing the squares”
Green function for diffusion equation, continued

- The result of the integral is actually the Green function $G(x, x'; t, t')$

$$G(x, x'; t, t') = \frac{1}{[4\pi \alpha^2 (t - t')]^{1/2}} e^{- (x-x')^2 / 4 \alpha^2 (t-t')}$$

- Notice that the Green function only depends on $x - x'$ and $t - t'$
- We find that at all times, $\int_{-\infty}^{\infty} G(x, x'; t, t') dx = 1$
- Then if we have a $t = 0$ distribution $u(x, t = 0)$, we can find $u(x, t)$ just by doing the integral,

$$u(x, t) = \int_{-\infty}^{\infty} G(x, x'; t, 0) u(x', t = 0) dx'$$