Laplace equation in Cartesian coordinates

- The Laplace equation is written

$$\nabla^2 \phi = 0$$

- For example, let us work in two dimensions so we have to find $\phi(x, y)$ from,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

- We use the method of *separation of variables* and write $\phi(x, y) = X(x)Y(y)$

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

Laplace equation in Cartesian coordinates, continued

• Again we have two terms that only depend on one independent variable, so

$$\frac{Y''}{Y} = -k^2$$

• This is called a Helmholtz equation (we've seen in before), and we can write it

$$Y'' + k^2 Y = 0$$

• The we have another equation to solve,

$$X'' - k^2 X = 0$$

• Here k is real and $k \geq 0$
• Could we have done this a different way? Yes!

Laplace equation in Cartesian coordinates, continued

• We could have a different sign for the constant, and then

$$Y'' - k^2 Y = 0$$

• The we have another equation to solve,

$$X'' + k^2 X = 0$$

• We will see that the choice will determine the nature of the solutions, which in turn will depend on the boundary conditions

Steady-state temperature in a semi-infinite plate

- Imagine a metal plate bounded at $y = 0$ but extending to infinity in the $+y$ direction, and from $x = 0$ to $x = 10$
- Hold the $y = 0$ surface at $T = 100$, and as $y \to +\infty$, $T = 0$
- Surface at $x = 0$ and $x = 10$ are both held at $T = 0$
- Find the scalar temperature field $T(x, y)$ inside the plate
- Again we solve $\nabla^2 T = 0$, or $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$
- We apply separation of variables $T(x, y) = X(x)Y(y)$, and choose the sign of the constant carefully!

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

Steady-state temperature in semi-infinite plate, continued

• We choose the sign of the constant to give us reasonable behavior based on the boundary conditions

$$X'' + k^2 X = 0$$

$$Y'' - k^2 Y = 0$$

• From the first ordinary differential equation, we get $X(x) = \sin kx$ and $X(x) = \cos kx$

• Since $T = 0$ at $x = 0$, the $\cos kx$ solution does not work

• Boundary condition $T = 0$ at $x = 10$ means we have solutions $X_n(x) = \sin \frac{n\pi x}{10}$, with $n = 1, 2, 3, \ldots$

Steady-state temperature in semi-infinite plate, continued

• Now the equation for $Y(y)$, *taking care that we now have* $k_n^2 = (\frac{n\pi}{10})^2$

$$Y'' - k_n^2 Y = 0$$

• We get solutions $Y_n(y) = e^{k_n y}$ and $Y_n(y) = e^{-k_n y}$
• Since $T = 0$ as $y \to \infty$, we have $Y_n(y) = e^{-k_n y} = e^{-n\pi y/10}$
• We can then write the general solutions that satisfy the boundary conditions as $y \to \infty$, and at $x = 0$ and $x = 10$

$$T(x, y) = \sum_{n=1}^{\infty} b_n e^{-n\pi y/10} \sin \frac{n\pi x}{10}$$

Steady-state temperature in semi-infinite plate, continued

- To determine b_n coefficients, use that $T = 100$ at $y = 0$

$$T(x, y = 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10} = 100$$

- A Fourier series! We find the b_n from

$$b_n = \frac{2}{10} \int_0^{10} 100 \sin \frac{n\pi x}{10} dx = (20) \left(\frac{10}{n\pi} \right) \left(- \cos \frac{n\pi x}{10} \right) |_0^{10}$$

- We find $b_n = \frac{400}{n\pi}$ for odd n, and $b_n = 0$ for even n

$$T = \frac{400}{\pi} \left(e^{-\pi y/10} \sin \frac{\pi x}{10} + \frac{1}{3} e^{-3\pi y/10} \sin \frac{3\pi x}{10} + \frac{1}{5} e^{-\pi y/2} \sin \frac{\pi x}{2} + ... \right)$$

Time-dependent diffusion or heat flow

- If $\nabla^2 T \neq 0$, then the temperature field becomes time-dependent
- Imagine a strip of metal, extending to $\pm\infty$ in the y direction, but bounded at $x = 0$ and $x = l$
- At $t = 0$, the system is in steady-state with $u = 0$ at $x = 0$, and $u = 100$ at $x = l$
- The temperature profile at $t = 0$ satisfies $\nabla^2 u = 0$, or $\frac{d^2 u}{dx^2} = 0$
- We find initially a temperature profile $u = 100\frac{x}{l}$
- Now abruptly set $u = 0$ at $x = 0$ and $x = l$
- The temperature profile is now time-dependent

$$\frac{d^2 u}{dx^2} = \frac{1}{\alpha^2}\frac{du}{dt}$$

Time-dependent diffusion or heat flow, continued

- We apply separation of variables $u(x,t) = T(t)X(x)$

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = -k^2$$

- We find $T = e^{-k^2\alpha^2 t}$
- From the boundary conditions $u = 0$ at $x = 0$ and $x = l$, we find $X = \sin\frac{n\pi x}{l}$ with $n = 1, 2, 3, \ldots$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} e^{-(\frac{n\pi x}{l})^2 t}$$

- We find the b_n from the initial condition $u(x, t = 0) = 100\frac{x}{l}$

Time-dependent diffusion or heat flow, initial conditions

• We can solve for the initial conditions ($t = 0$)

$$u(x, t = 0) = \frac{100}{l} x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

• Notice compared to before, when we expanded periodic functions, the $y_0(x)$ and $v_0(x)$ are *not* periodic with period l
• However, we can show that the $\sin \frac{n\pi x}{l}$ make a complete, orthogonal set over the interval $0 < x < l$
• We find the b_n then, using that

$$\int_0^l \sin \frac{m\pi x}{l} \sin \frac{n\pi x}{l} dx = \frac{l}{2} \delta_{m,n}$$

• We find that $b_n = \frac{200}{\pi} \frac{(-1)^{n-1}}{n}$

Time-dependent diffusion or heat flow, final answer

- Then we can finally write the series solution

$$u = \frac{200}{\pi}\left[e^{-(\pi\alpha/l)^2 t}\sin\frac{\pi x}{l} - e^{-(2\pi\alpha/l)^2 t}\sin\frac{2\pi x}{l} + e^{-(3\pi\alpha/l)^2 t}\sin\frac{3\pi x}{l} - \ldots\right.$$

- Notice that as $t \to \infty$, we reach equilibrium $T = 0$ everywhere
- Another example, slightly different... final conditions involve a temperature gradient