Expansion of $1/r$ potential in Legendre polynomials

- In electrostatics and gravitation, we see scalar potentials of the form $V = \frac{K}{d}$
- Take $d = |\vec{R} - \vec{r}| = \sqrt{R^2 - 2Rr\cos\theta + r^2} = R\sqrt{1 - 2\frac{r}{R}\cos\theta + (\frac{r}{R})^2}$
- Use $h = \frac{r}{R}$ and $x = \cos\theta$, and then we see we have the generating function!

$$V = \frac{K}{R}\left(1 - 2hx + h^2\right)^{-1/2} = \frac{K}{R}\sum_{l=1}^{\infty} h^l P_l(x)$$

- Then in terms of the r and θ variables, we have

$$V = K\sum_{l=0}^{\infty} \frac{r^l P_l(\cos\theta)}{R^{l+1}}$$

Multipole expansion

- If we have make charges q_i at different coordinates $\vec{r}_i$, then we can use this to find the electrostatic potential at $\vec{R}$

$$V = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{\sum_i q_i r_i^l P_l(\cos\theta_i)}{R^{l+1}}$$

- Or if we have a continuous distribution $\rho(\vec{r})$,

$$V = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{\int\int\int r^l P_l(\cos\theta)\rho d\tau}{R^{l+1}}$$

- Lowest order term $l = 0$, is just the total charge, $V \propto \frac{1}{R}$

$$Q = \int\int\int \rho d\tau$$

Multipole expansion, continued

- Next order term $l = 1$ is the dipole moment, $V \propto \frac{1}{R^2}$

$$p = \int\int\int r \cos\theta \rho d\tau$$

- Writing both the $l = 0$ (monopole) and $l = 1$ (dipole) terms, we have

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{R} + \frac{p}{R^2} + ...\right]$$

- Higher order terms take into account more details of the distribution with contributions that fall off faster with increasing R
- For example, the quadrupole moments contribute a potential $\propto \frac{1}{R^3}$

Associated Legendre equation

$$(1-x^2)y'' - 2xy' + \left[l(l+1) - \frac{m^2}{1-x^2} \right] y = 0$$

- The Legendre equation corresponds to $m = 0$
- We again have l and m integer, and write the solutions $P_l^m(x)$
- The $P_l^m(x)$ can be found from the $P_l(x)$ using, for positive m

$$P_l^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x)$$

• Since the associated Legendre equation is the same for positive and negative m, $P_l^{-m}(x) = P_l^m(x)$

• Using the fact that the highest power of x in $P_l(x)$ is x^l, we have then $-l \leq m \leq l$, and m and l are both integers

Orthogonality of the associated Legendre functions

- The associated Legendre functions $P_l^m(x)$ are orthogonal on the interval $-1 < x < 1$ for each value of m

$$\int_{-1}^{1} P_l^m(x) P_{l'}^m(x) dx = \delta_{l,l'} \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!}$$

- We can still make an expansion in these polynomials for $m \neq 0$

$$f(x) = \sum_{l=|m|}^{\infty} c_l P_l^m(x)$$

Connection to Laplacian in spherical coordinates (Chapter 13)

• We might often encounter the Laplace equation and spherical coordinates might be the most convenient

$$\nabla^2 u(r,\theta,\phi) = 0$$

• We already saw in Chapter 10 how to write the Laplacian operator in spherical coordinates,

$$\nabla^2 \phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}$$

• This is a partial differential equation we will solve by what will become a standard approach of *separation of variables*
• We take $u(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$, which we substitute in and then multipy by $\frac{r^2}{R\Theta\Phi}$

Separation of variables for the Laplace equation

- Because of this separation we now wind up with total derivatives

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{1}{\Phi}\frac{1}{\sin^2\theta}\frac{d^2\Phi}{d\phi^2}=0$$

- Because the first two terms do not depend on ϕ, we must have from the last term

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=-m^2$$

- The functions $\Phi(\phi)$ must be periodic with period 2π, and this suggest that m is an integer and $\Phi=\sin m\phi$ or $\Phi=\cos m\phi$

Separation of variables for the Laplace equation, continued

- Now we can replace $\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}$ with the constant $-m^2$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)-\frac{m^2}{\sin^2\theta}=0$$

- The first term is a function of only r, and the last two terms are now functions of only θ, so we can take

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=k$$

- Here k is just a constant that we will later take $k=l(l+1)$
- Then we have the final equation for the θ-dependent terms,

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left(k-\frac{m^2}{\sin^2\theta}\right)\Theta=0$$

Separation of variables for the Laplace equation, continued

- We solved for Φ, but we still need to solve for R and Θ
- For the moment, let's focus only on the Θ function that solves

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left(k-\frac{m^2}{\sin^2\theta}\right)\Theta=0$$

- Make a change of variables to $x=\cos\theta$ and then $\Theta(\theta)\to y(x)$
- Then we have $\frac{d}{d\theta}=\frac{dx}{d\theta}\frac{d}{dx}=-\sin\theta\frac{d}{dx}$, and $\frac{1}{\sin^2\theta}=\frac{1}{1-x^2}$
- We can then obtain the associated Legendre equation

$$y''-2xy'+\left[l(l+1)-\frac{m^2}{1-x^2}\right]y=0$$

- So we have found the associated Legendre equation from Laplace equation in spherical coordinates!
- Hence we know $\Theta(\theta)=P_l^m(\cos\theta)$

Laplace equation in spherical coordinates, continued

- We will see later that $R(r) = r^l$ or $R(r) = r^{-l-1}$
- Recall, we found $\Phi(\phi) = \cos(m\phi)$ or $\Phi(\phi) = \sin(m\phi)$
- Finally $u(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$
- So our solutions are $u = r^l P_l^m(\cos\theta)\sin m\phi$, $u = r^l P_l^m(\cos\theta)\cos m\phi$, $u = r^{-l-1}P_l^m(\cos\theta)\sin m\phi$, $u = r^{-l-1}P_l^m(\cos\theta)\cos m\phi$
- Superposition applies! So in general, a solution might be a linear combination of these solutions for different l and m
- Will depend on boundary conditions! For example, maybe we are interested in solutions near $r = 0$ where r^{-l-1} diverges, then

$$u(r, \theta, \phi) = \sum_{m=-l}^{l} \sum_{l=0}^{\infty} r^l P_l^m(\cos\theta)\left[a_{lm}\cos m\phi + b_{lm}\sin m\phi\right]$$

Steady-state temperature in a sphere

- Heat travels by diffusion, so we have

$$\nabla^2 T = \frac{1}{\alpha^2}\frac{\partial T}{\partial t}$$

- The constant $\frac{1}{\alpha^2}$ depends on the properties of the medium, in one can show that $\frac{1}{\alpha^2} = \frac{C}{\kappa}$, where C is the specific heat capacity and κ is thermal conductivity
- This can be derived from Fourier's Law $\vec{J} = -\kappa \vec{\nabla} T$ (For an isotropic medium, actually the conductivity is a rank 2 tensor!) and the continuity equation $\frac{\partial u}{\partial t} = -\vec{\nabla} \cdot \vec{J} = \kappa \nabla^2 T$ and $\frac{\partial u}{\partial t} = C \frac{\partial T}{\partial t}$
- In steady state $\frac{\partial T}{\partial t} = 0$, so $\vec{\nabla} T = 0$

Steady-state temperature in a sphere

• Consider a sphere of radius $r = 1$, with the temperature $T = 100$ on the top half ($z > 0$ or $0 < \theta < \pi/2$) and $T = 0$ on the bottom half ($z < 0$ or $\pi/2 < \theta < \pi$)

• We know that our solution is a solution to Laplace equation $\nabla^2 T = 0$ most conveniently in spherical coordinates

$$T(r, \theta, \phi) = \sum_{m=-l}^{l} \sum_{l=0}^{\infty} r^l P_l^m(\cos\theta) \left[a_{lm} \cos m\phi + b_{lm} \sin m\phi\right]$$

• Based on the problem, we see there is no ϕ dependence, so we only require $m = 0$ and $\cos m\phi = 1$, so we simplify

$$T(r, \theta) = \sum_{l=0}^{\infty} c_l r^l P_l^{m=0}(\cos\theta)$$