Vector operators in curvilinear coordinate systems

• In a Cartesian system, take $x_1 = x$, $x_2 = y$, and $x_3 = z$, then an element of arc length ds^2 is,

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2$$

• In a general system of coordinates, we still have x_1, x_2, and x_3
• For example, in cylindrical coordinates, we have $x_1 = r$, $x_2 = \theta$, and $x_3 = z$
• We have already shown how we can write ds^2 in cylindrical coordinates,

$$ds^2 = dr^2 + r^2 d\theta + dz^2 = dx_1^2 + x_1^2 dx_2^2 + dx_3^2$$

• We write this in a general form, with h_i being the *scale factors*

$$ds^2 = h_1^2 dx_1^2 + h_2^2 dx_2^2 + h_3^2 dx_3^2$$

• We see then for cylindrical coordinates, $h_1 = 1$, $h_2 = r$, and $h_3 = 1$

Curvilinear coordinates

- For an vector displacement $\vec{ds}$

$$\vec{ds} = \hat{e}_1 h_1 dx_1 + \hat{e}_2 h_2 dx_2 + \hat{e}_3 h_3 dx_3$$

- Back to our example of cylindrical coordiantes, $\hat{e}_1 = \hat{e}_r$, $\hat{e}_2 = \hat{e}_\theta$, and $\hat{e}_3 = \hat{e}_z$, and

$$\vec{ds} = \hat{e}_r dr + \hat{e}_\theta r d\theta + \hat{e}_z dz$$

- These are orthogonal systems, but it would not have to be!

$$ds^2 = \sum_{i=1}^{3} \sum_{j=1}^{3} g_{ij} dx_i dx_i$$

- The g_{ij} is the metric tensor, and for an orthogonal system it is diagonal with $g_i = h_i^2$

Vector operators in general curvilinear coordinates

• Recall the *directional derivative* $\frac{d\phi}{ds}$ along $\vec{u}$, where $\vec{u}$ was a unit vector

$$\frac{d\phi}{ds} = \nabla\phi \cdot \vec{u}$$

• Now the $\vec{u}$ becomes the unit vectors in an orthogonal system, for example in cylindrical coordinates
• Now we recall that $ds^2 = ds^2 = h_1^2 dx_1^2 + h_2^2 dx_2^2 + h_3^2 dx_3^2$
• Let's take a cylindrical system, first consider $\vec{u} = \hat{e}_r$, then $ds = dr$

$$\vec{\nabla}\phi(r,\theta,z) \cdot \hat{e}_r = \frac{\partial\phi}{\partial r}$$

Vector operators in general curvilinear coordinates

- Next $\vec{u} = \hat{e}_{r\theta}$, then $ds = rd\theta$ $(h_2 = r)$

$$\vec{\nabla}\phi(r,\theta,z)\cdot\hat{e}_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta}$$

- It is also easy to show,

$$\vec{\nabla}\phi(r,\theta,z)\cdot\hat{e}_z = \frac{\partial\phi}{\partial z}$$

- Now that we have the projections, we can find $\vec{\nabla}\phi$ in cylindrical coordinates,

$$\vec{\nabla}\phi = \frac{\partial\phi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{e}_\theta + \frac{\partial\phi}{\partial z}\hat{e}_z$$

Gradient in curvilinear (orthogonal) coordinate system

- Most generally, we have

$$\vec{\nabla}\phi = \sum_{i=1}^{3} \hat{e}_i \frac{1}{h_i}\frac{\partial \phi}{\partial x_i}$$

- In Cartesian, obviously $h_1 = h_2 = h_3$, and $x_1 = x$, $x_2 = y$, and $x_3 = z$,

$$\vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

- In a spherical coordinate system $x_1 = r$, $x_2 = \theta$, and $x_3 = \phi$, then $h_1 = 1$, $h_2 = r$, and $h_3 = r\sin\theta$

$$\vec{\nabla}u = \frac{\partial u}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial u}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial u}{\partial \phi}\hat{e}_\phi$$

Divergence in curvilinear coordinates

• We recall in Cartesian coordinates, with $\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$, and the gradient operator $\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

$$\vec{\nabla} \cdot \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

• In a general orthogonal system, $\vec{V} = V_1\hat{e}_1 + V_2\hat{e}_2 + V_3\hat{e}_3$

• The difficultly comes because the unit vectors in a general orthogonal coordinate system may not be fixed

Divergence in curvilinear coordinates, continued

- First show that $\vec{\nabla} \cdot \left(\frac{\hat{e}_3}{h_1 h_2} \right) = 0$ (Problem 1)
- Assume $\hat{e}_1 \times \hat{e}_2 = \hat{e}_3$ (orthogonal coordinate system), and then obviously $\nabla x_1 = \frac{\hat{e}_1}{h_1}$ and $\nabla x_2 = \frac{\hat{e}_2}{h_2}$, and $\vec{\nabla} x_1 \times \vec{\nabla} x_2 = \frac{\hat{e}_3}{h_1 h_2}$, and next

$$\vec{\nabla} \cdot \left(\frac{\hat{e}_3}{h_1 h_2} \right) = \vec{\nabla} \cdot \left(\vec{\nabla} x_1 \times \vec{\nabla} x_2 \right)$$

- The vector relations at the end of Chapter 6 help to work out the right-hand side,

$$\vec{\nabla} \cdot \left(\vec{\nabla} x_1 \times \vec{\nabla} x_2 \right) = \vec{\nabla} x_2 \cdot \left(\vec{\nabla} \times \vec{\nabla} x_1 \right) - \vec{\nabla} x_1 \cdot \left(\vec{\nabla} \times \vec{\nabla} x_2 \right)$$

- But we have $\vec{\nabla} \times \vec{\nabla} x_1 = \vec{\nabla} \times \vec{\nabla} x_2 = 0$, so we have shown $\vec{\nabla} \cdot \left(\frac{\hat{e}_3}{h_1 h_2} \right) = 0$

Divergence in curvilinear coordinates, continued

- We use then that $\vec{\nabla} \cdot \left(\frac{\hat{e}_3}{h_1 h_2}\right) = 0$, and also $\vec{\nabla} \cdot \left(\frac{\hat{e}_1}{h_2 h_3}\right) = 0$ and $\vec{\nabla} \cdot \left(\frac{\hat{e}_2}{h_1 h_3}\right) = 0$

$$\vec{\nabla} \cdot \vec{V} = \vec{\nabla} \cdot \left(\frac{\hat{e}_1}{h_2 h_3} h_2 h_3 V_1 + \frac{\hat{e}_2}{h_1 h_3} h_1 h_3 V_2 + \frac{\hat{e}_3}{h_1 h_2} h_1 h_2 V_3 \right)$$

- We use then $\vec{\nabla} \cdot (\phi \vec{V}) = \vec{V} \cdot \vec{\nabla} \phi + \phi \vec{\nabla} \cdot \vec{V}$

$$\vec{\nabla} \cdot \vec{V} = \frac{\hat{e}_1}{h_2 h_3} \cdot \vec{\nabla}(h_2 h_3 V_1) + \frac{\hat{e}_2}{h_1 h_3} \cdot \vec{\nabla}(h_1 h_3 V_2) + \frac{\hat{e}_3}{h_1 h_2} \cdot \vec{\nabla}(h_1 h_2 V_3)$$

- Then we see that $\hat{e}_1 \cdot \vec{\nabla}(h_2 h_3 V_1) = \frac{1}{h_1} \frac{\partial}{\partial x_1}(h_2 h_3 V_1)$,etc.

Divergence in curvilinear coordinates, final result!

- Finally we get,

$$\vec{\nabla} \cdot \vec{V} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial x_1} (h_2 h_3 V_1) + \frac{\partial}{\partial x_2} (h_1 h_3 V_2) + \frac{\partial}{\partial x_3} (h_1 h_2 V_3) \right]$$

- Example: Cylindrical coordinates, $x_1 = r$, $x_2 = \theta$, and $x_3 = z$, with $h_1 = 1$, $h_2 = r$, and $h_3 = 1$
- In cylindrical coordinates, $\vec{V} = V_r \hat{e}_r + V_\theta \hat{e}_\theta + V_z \hat{e}_z$

$$\vec{\nabla} \cdot \vec{V} = \frac{1}{r} \left[\frac{\partial}{\partial r} (r V_r) + \frac{\partial}{\partial \theta} (V_\theta) + \frac{\partial}{\partial z} (r V_z) \right]$$

- Finally we simplify,

$$\vec{\nabla} \cdot \vec{V} = \frac{1}{r} \frac{\partial}{\partial r} (r V_r) + \frac{1}{r} \frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z}$$

Another important example: Divergence in spherical coordinates

- In spherical coordinates $\vec{V} = V_r \hat{e}_r + +V_\theta \hat{e}_\theta + V_\phi \hat{e}_\phi$
- Here we have $x_1 = r$, $x_2 = \theta$, and $x_3 = \phi$
- The scale factors are $h_1 = 1$, $h_2 = r$, and $h_3 = r \sin\theta$

$$\vec{\nabla} \cdot \vec{V} = \frac{1}{r^2 \sin\theta} \left[\frac{\partial}{\partial r} \left(r^2 \sin\theta V_r \right) + \frac{\partial}{\partial \theta} \left(r \sin\theta V_\theta \right) + \frac{\partial}{\partial \phi} \left(r V_\phi \right) + \right]$$

- This simplifies to

$$\vec{\nabla} \cdot \vec{V} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 V_r \right) + \frac{1}{r} \frac{\partial V_\theta}{\partial \theta} + \frac{1}{r \sin\theta} \frac{\partial V_\phi}{\partial \phi}$$

Laplacian

- We want to have an expression for $\nabla^2 u$ for a general curvilinear system
- We start with $\vec{\nabla} u = \frac{\hat{e}_1}{h_1}\frac{\partial u}{\partial x_1} + \frac{\hat{e}_2}{h_2}\frac{\partial u}{\partial x_2} + \frac{\hat{e}_3}{h_3}\frac{\partial u}{\partial x_3}$
- Then with $\vec{V} = V_1\hat{e}_1 + V_2\hat{e}_2 + V_3\hat{e}_3$, we have $V_1 = \frac{1}{h_1}\frac{\partial u}{\partial x_1}$ and $V_2 = \frac{1}{h_2}\frac{\partial u}{\partial x_2}$ and $V_3 = \frac{1}{h_3}\frac{\partial u}{\partial x_3}$
- Now we go back to our formula for the divergence,

$$\vec{\nabla} \cdot \vec{V} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}(h_2 h_3 V_1) + \frac{\partial}{\partial x_2}(h_1 h_3 V_2) + \frac{\partial}{\partial x_3}(h_1 h_2 V_3)\right]$$

- The $\nabla^2 u = \vec{\nabla} \cdot \vec{\nabla} u$ is

$$\nabla^2 u = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial u}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial u}{\partial x_3}\right)\right]$$

Example: $\nabla^2 u$ in cylindrical coordinates

$$\nabla^2 u = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial u}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial u}{\partial x_3}\right)\right.$$

- In cylindrical coordinates we have $x_1 = r$, $x_2 = \theta$, and $x_3 = z$, with $h_1 = 1$, $h_2 = r$, and $h_3 = 1$

$$\nabla^2 u = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial u}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial u}{\partial z}\right)\right]$$

- This simplifies to

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$

Example: $\nabla^2 u$ in spherical coordinates

$$\nabla^2 u = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial u}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial u}{\partial x_3}\right)\right.$$

- In spherical coordinates we have $x_1 = r$, $x_2 = \theta$, and $x_3 = \phi$, with $h_1 = 1$, $h_2 = r$, and $h_3 = r\sin\theta$

$$\nabla^2 u = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\sin\theta}\frac{\partial u}{\partial \phi}\right)\right]$$

- This simplifies to

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}$$

Curl in curvilinear coordinates

- As before we work with some general system and vector $\vec{V} = V_1\hat{e}_1 + V_2\hat{e}_2 + V_3\hat{e}_3$
- Derivation following problem 2, start with $\vec{\nabla}x_1 = \frac{\hat{e}_1}{h_1}$, and then $\vec{\nabla} \times \vec{\nabla}x_1 = 0$
- Hence we see $\vec{\nabla} \times \frac{\hat{e}_1}{h_1} = \vec{\nabla} \times \frac{\hat{e}_2}{h_2} = \vec{\nabla} \times \frac{\hat{e}_3}{h_3} = 0$
- Write $\vec{V} = \frac{\hat{e}_1}{h_1}(h_1 V_1) + \frac{\hat{e}_2}{h_2}(h_2 V_2) + \frac{\hat{e}_3}{h_3}(h_3 V_3)$
- Now we use the relation $\vec{\nabla} \times (\phi\vec{U}) = \phi(\vec{\nabla} \times \vec{U}) - \vec{U} \times (\vec{\nabla}\phi)$

$$\vec{\nabla} \times \vec{V} = -\frac{\hat{e}_1}{h_1} \times \vec{\nabla}(h_1 V_1) - \frac{\hat{e}_2}{h_2} \times \vec{\nabla}(h_2 V_2) - \frac{\hat{e}_3}{h_3} \times \vec{\nabla}(h_3 V_3)$$

- Consider just the first term, to keep it simple,

$$-\frac{\hat{e}_1}{h_1} \times \vec{\nabla}(h_1 V_1) = -\frac{\hat{e}_1}{h_1} \times \left[\frac{\hat{e}_1}{h_1}\frac{\partial(h_1 V_1)}{\partial x_1} + \frac{\hat{e}_2}{h_2}\frac{\partial(h_1 V_1)}{\partial x_2} + \frac{\hat{e}_3}{h_3}\frac{\partial(h_1 V_1)}{\partial x_3}\right]$$

Curl in curvilinear coordinates, continued

- Next we just use $\hat{e}_1 \times \hat{e}_1 = 0$, $\hat{e}_1 \times \hat{e}_2 = \hat{e}_3$, and $\hat{e}_1 \times \hat{e}_3 = -\hat{e}_2$

$$-\frac{\hat{e}_1}{h_1} \times \vec{\nabla}(h_1 V_1) = -\frac{1}{h_1 h_2 h_3}\left[h_3 \hat{e}_3 \frac{\partial(h_1 V_1)}{\partial x_2} - h_2 \hat{e}_2 \frac{\partial(h_1 V_1)}{\partial x_3}\right]$$

- We can do the other terms as well, and the final result is expressed as a determinant

$$\vec{\nabla} \times \vec{V} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix}$$

Example: Curl in cylindrical coordinates

$$\vec{\nabla} \times \vec{V} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix}$$

• In cylindrical coordinates we have $x_1 = r$, $x_2 = \theta$, and $x_3 = z$, with $h_1 = 1$, $h_2 = r$, and $h_3 = 1$

$$\vec{\nabla} \times \vec{V} = \frac{1}{r} \begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ V_r & rV_\theta & V_z \end{vmatrix}$$

• We can evaluate the determinant

$$\vec{\nabla} \times \vec{V} = \left(\frac{1}{r} \frac{\partial V_z}{\partial \theta} - \frac{\partial V_\theta}{\partial z} \right) \hat{e}_r + \left(\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r} \right) \hat{e}_\theta + \frac{1}{r} \left(\frac{\partial}{\partial r}(rV_\theta) - \frac{\partial V_r}{\partial \theta} \right) \hat{e}$$

Example: Curl in spherical coordinate

$$\vec{\nabla} \times \vec{V} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix}$$

- In spherical coordinates we have $x_1 = r$, $x_2 = \phi$, and $x_3 = \theta$, with $h_1 = 1$, $h_2 = r \sin\theta$, and $h_3 = r$

$$\vec{\nabla} \times \vec{V} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r \sin\theta \hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ V_r & rV_\theta & r \sin\theta V_\phi \end{vmatrix}$$

- We can evaluate the determinant to get $\vec{\nabla} \times \vec{V}$