Chapter 2: Complex numbers

Complex numbers are commonplace in physics and engineering. In particular, complex numbers enable us to simplify equations and/or more easily find solutions to equations. We will explore the damped, driven simple-harmonic oscillator as an example of the use of complex numbers.

By the end of this chapter you should be able to...

- Represent complex numbers in various ways
- Use complex algebra
- Complex infinite series
- Determine functions of complex numbers
- Use Euler’s formula
- Use exponential and trigonometric functions
- Define and use hyperbolic functions
- Use logarithms
- Do all of the above with complex numbers!
- Solve harmonic oscillator and driven-damped oscillator
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Introduction to Theoretical Methods
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Real and imaginary parts of a complex number

Consider a complex number written as \( z = x + iy \)

\[ \text{Re } z = x \]

What if it is not presented as \( z = x + iy \)? We can always simplify to this form with \textit{complex algebra}.

\[
(2 + i)^2 = 4 + 4i - 1 = 3 + 4i \tag{1}
\]

What if there is a complex denominator? We can multiply in the numerator and denominator by the \textit{complex conjugate} or \textit{conjugate} of the denominator.

\[
\frac{i + 1}{i - 1} = \left( \frac{i + 1}{i - 1} \right) \left( \frac{-i - 1}{-i - 1} \right) = -i \tag{2}
\]
Consider a complex number written as $z = x + iy$

- $\text{Re } z = x$
- $\text{Im } z = y$

What if it is not presented as $z = x + iy$? We can always simplify to this form with complex algebra.

$$ (2 + i)^2 = 4 + 4i - 1 = 3 + 4i \quad (1) $$

What if there is a complex denominator? We can multiply in the numerator and denominator by the complex conjugate or conjugate of the denominator.

$$ \frac{i + 1}{i - 1} = \left( \frac{i + 1}{i - 1} \right) \left( \frac{-i - 1}{-i - 1} \right) = -i \quad (2) $$
Representing a complex number as \( z = x + iy \) suggests we can plot them in a complex plane.
The complex plane

- Representing a complex number as \( z = x + iy \) suggests we can plot them in a \textit{complex plane}.
- We can also represent in \textit{polar form} \( z = re^{i\theta} \).
The complex plane

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- We can also represent in polar form \( z = re^{i\theta} \)
- Equivalently \( z = r (\cos \theta + i \sin \theta) \)
The complex plane

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- We can also represent in polar form \( z = re^{i\theta} \)
- Equivalently \( z = r (\cos \theta + i \sin \theta) \)
- The modulus or magnitude is \( r \)
The complex plane

- Representing a complex number as $z = x + iy$ suggests we can plot them in a complex plane.
- We can also represent in polar form $z = re^{i\theta}$
- Equivalently $z = r (\cos \theta + i \sin \theta)$
- The *modulus* or *magnitude* is $r$
- The *angle* or *phase* is $\theta$
We need to work with complex representations
\[ z = x + iy = re^{i\theta} = r (\cos \theta + i \sin \theta) \]. It should become easy to go from one representation to another!

- For \( z = x + iy \), the complex conjugate \( z^* = x - iy \)
We need to work with complex representations

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- For \( z = re^{i\theta} \), \( z^* = re^{-i\theta} \)
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- For \( z = re^{i\theta} \), \( z^* = re^{-i\theta} \)
- Given \( z = x + iy \), we can find \( r \) from
  \[ r = |z| = \sqrt{z^*z} = \sqrt{x^2 + y^2} \]
Complex algebra

We need to work with complex representations

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- Given \( z = x + iy \), we see \( \cos \theta = x/r \) and \( \sin \theta = y/r \)
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\[ z = x + iy = re^{i\theta} = r(\cos \theta + i \sin \theta) \]. It should become easy to go from one representation to another!

- For \( z = x + iy \), the complex conjugate \( z^* = x - iy \)
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- Given \( z = x + iy \), we can find \( r \) from 
  \[ r = |z| = \sqrt{z^*z} = \sqrt{x^2 + y^2} \]
- Given \( z = x + iy \), we see \( \cos \theta = x/r \) and \( \sin \theta = y/r \)
- We can find \( \theta \) using \( \theta = \cos^{-1} x/r = \sin^{-1} y/r \)
- Also \( \tan \theta = \frac{y}{x} \), or \( \theta = \tan^{-1} \frac{y}{x} \)
Complex power series

Previously we had real power series

$$\sum_{n=1}^{\infty} a_n x^n$$  \hspace{1cm} (3)

We can also consider a power series in $z = x + iy$

$$\sum_{n=1}^{\infty} a_n z^n$$  \hspace{1cm} (4)

We need to determine whether a series converges using the ratio test:

$$\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}z^{n+1}}{a_n z^n} \right|$$  \hspace{1cm} (5)

When $\rho < 1$, the series converges.
Consider the series,

\[ 1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + ... \]  

(6)

We see from this that,

\[ \rho = \lim_{n \to \infty} \left| \frac{a_{n+1}z^{n+1}}{a_n z^n} \right| = \lim_{n \to \infty} \left| \frac{iz}{n + 1} \right| = 0 \]  

(7)

So indeed this series converges. In fact we will see that it converges to \( e^{iz} \).
Euler’s formula

We previously had the infinite-series (i.e. Mclaurin series) for $\cos \theta$ and $\sin \theta$,

\[
\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots \quad (8)
\]

\[
\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots \quad (9)
\]

Using $\frac{de^z}{dz} = e^z$, we can also represent $e^z$ by the series,

\[
e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \ldots \quad (10)
\]

If we take $z = i\theta$, then comparison of these series proves Euler’s formula,

\[
e^{i\theta} = \cos \theta + i \sin \theta
\]
Examples using Euler’s formula

• Express \( z = 2e^{\frac{i\pi}{4}} \) in the form \( z = x + iy \)
  From Euler’s formula,
  
  \[ 2e^{\frac{i\pi}{4}} = 2 \cos \frac{\pi}{4} + 2i \sin \frac{\pi}{4} = \sqrt{2} + i\sqrt{2} \]

• Express \( z = \left(\frac{i\sqrt{2}}{1+i}\right)^6 \) in the form \( z = x + iy \)
  Using Euler’s formula, \( i = e^{\frac{i\pi}{2}} \) and \( 1 + i = \sqrt{2}e^{\frac{\pi i}{4}} \), then we see
  
  \[ \left(\frac{i\sqrt{2}}{1+i}\right)^6 = \left(e^{\frac{i\pi}{2}}\right)^6 = \left(e^{\frac{3\pi i}{4}}\right)^6 = e^{\frac{3\pi i}{2}} \]

  Then we use Euler’s formula,
  
  \[ e^{\frac{3\pi i}{2}} = \cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} = -i \]
Powers and roots of complex numbers

We can start with the form $z = re^{i\theta}$, then to take to the $n$th power,

$$z^n = \left( re^{i\theta} \right)^n = r^n e^{in\theta}$$

Likewise, if we want the $n$th root of $z$,

$$z^{1/n} = \left( re^{i\theta} \right)^{1/n} = r^{1/n} e^{i\theta/n}$$

We used this in the second example in the last slide.
Examples of roots of complex numbers

- For \( z = -8 \), determine \( z^{1/3} = (-8)^{1/3} \) in the form \( x + iy \)

  For \( z = -8 \), we can see \( r = 8 \) and \( \theta = \pi \), so \( z = 8e^{i\pi} \), and then

  \[
  z^{1/3} = (-8)^{1/3} = (8e^{i\pi})^{1/3} = 2e^{i\pi/3}
  \]

  Then we use Euler’s formula,

  \[
  2e^{i\pi/3} = 2 \cos \pi/3 + 2i \sin \pi/3 = 1 + i\sqrt{3}
  \]

  This can be easily checked without invoking Euler’s formula,

  \[
  (1 + i\sqrt{3})^3 = -8
  \]
Euler’s formula can be used to find representations of $\cos \theta$ and $\sin \theta$.

\[
\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \quad (11)
\]

\[
\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad (12)
\]

Instead of just real $\theta$, this also applies for complex $z$,

\[
\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad (13)
\]

\[
\cos z = \frac{e^{1z} + e^{-iz}}{2} \quad (14)
\]
Complex exponentials are useful for integrating products of sin and cos functions. For example

- Solve the integral \( \int_{-\pi}^{\pi} \cos 2x \cos 3x \, dx \).

First we make note that \( \cos 2x = \frac{e^{2ix} + e^{-2ix}}{2} \) and \( \cos 3x = \frac{e^{3ix} + e^{-3ix}}{2} \)

\[
\int_{-\pi}^{\pi} \cos 2x \cos 3x \, dx = \frac{1}{4} \int_{-\pi}^{\pi} \left( e^{5ix} + e^{ix} + e^{-ix} + e^{-5ix} \right) \, dx
\]

This integral is easy, and we get

\[
\frac{1}{4} \left[ \left( \frac{e^{5ix} - e^{-5ix}}{5i} \right) + \left( \frac{e^{ix} - e^{-ix}}{i} \right) \right]_{-\pi}^{\pi} = \left[ \frac{1}{10} \sin 5x + \frac{1}{2} \sin x \right]_{-\pi}^{\pi} = 0
\]

The complex exponential form is also useful in differential equations.
Hyperbolic functions

If we start with our representations of cos and sin as complex exponentials, then consider pure imaginary argument (e.g. $z = iy$)

\[
\sin iy = i \frac{e^y - e^{-y}}{2} \tag{15}
\]

\[
\cos iy = \frac{e^y + e^{-y}}{2} \tag{16}
\]

This provides us with definitions for the hyperbolic functions, \(\sinh y = -i \sin iy\) and \(\cosh y = \cos iy\). More generally for any \(z\),

\[
\sinh z = \frac{e^z - e^{-z}}{2} \tag{17}
\]

\[
\cosh z = \frac{e^z + e^{-z}}{2} \tag{18}
\]

Also \(\tanh z = \frac{\sinh z}{\cosh z}\), \(\coth z = \frac{\cosh z}{\sinh z}\), etc.
Example with hyperbolic functions

• Write $\sinh \left( \ln 2 + \frac{i\pi}{3} \right)$ in $x + iy$ form

We use the representation of $\sinh$ in terms of exponentials,

$$\sinh \left( \ln 2 + \frac{i\pi}{3} \right) = \frac{e^{(\ln 2 + i\pi/3)} - e^{-(\ln 2 + i\pi/3)}}{2} = \frac{2e^{i\pi/3} - (1/2)e^{-i\pi/3}}{2}$$

Then using Euler’s formula for the complex exponentials, we get

$$\sinh \left( \ln 2 + \frac{i\pi}{3} \right) = \frac{3}{8} + \frac{5\sqrt{3}}{8}i$$
Logarithms of complex numbers

- It is possible to take logarithms of negative or even complex numbers

\[
\ln(re^{i\theta}) = \ln r + i(\theta \pm 2n\pi)
\]  

(19)
Logarithms of complex numbers

- It is possible to take logarithms of negative or even complex numbers.
- If $z = e^w$ then $w = \ln z$ where $z$ is complex.

$$\ln (re^{i\theta}) = \ln r + i(\theta \pm 2n\pi)$$  \hspace{1cm} (19)
Logarithms of complex numbers

- It is possible to take logarithms of negative or even complex numbers
- If $z = e^w$ then $w = \ln z$ where $z$ is complex
- $w = \ln z = \ln(re^{i\theta}) = \ln r + i\theta$

\[ \ln (re^{i\theta}) = \ln r + i(\theta \pm 2n\pi) \quad (19) \]
It is possible to take logarithms of negative or even complex numbers.

If $z = e^w$ then $w = \ln z$ where $z$ is complex.

$w = \ln z = \ln(re^{i\theta}) = \ln r + i\theta$

Since we can add $2n\pi$ to $\theta$, $n$ integer, and get same result, we have most generally:

$$\ln (re^{i\theta}) = \ln r + i(\theta \pm 2n\pi) \quad (19)$$
We can take a complex number $a$ to a complex power $b$! We can often evaluate using,

$$a^b = e^{b \ln a} \quad (20)$$

- For example, evaluate $i^i$ in the form $x + iy$

$$i^i = e^{i \ln i}$$

Then using $i = e^{i\pi/2} e^{\pm 2n\pi i}$ (from Euler’s formula), we see $i \ln i = -\pi/2 \pm 2n\pi$, and finally,

$$i^i = e^{-\pi/2 \pm 2n\pi}$$

While there are an infinite number of answers, note that they are all real!
• Evaluate $i^{1/2}$ in the form $x + iy$.

$$i^{1/2} = e^{(1/2)\ln i} = e^{(1/2)\ln (e^{i\pi/2 \pm 2\pi n})} = e^{i\pi/4 \pm in\pi}$$

Since $e^{in\pi} = 1$ for even $n$ and $e^{in\pi} = -1$ for odd $n$, we have two answers,

$$i^{1/2} = \pm e^{i\pi/4} = \pm \frac{1 + i}{\sqrt{2}}$$

Not surprising that the square root gives two possible results, as it does for real numbers.

• Check directly this result,

$$i^{1/2}i^{1/2} = \left[ \frac{1 + i}{\sqrt{2}} \right] \left[ \frac{1 + i}{\sqrt{2}} \right] = i$$
For $w = \cos z$, we have $z = \arccos w = \cos^{-1} w$

It is convenient to use the forms,

$$w = \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$w = \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$
For $w = \cos z$, we have $z = \arccos w = \cos^{-1} w$.

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If $z$ is real, $w$ is always between $-1$ and $+1$.

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Likewise $w = \sin z$, we have $z = \arcsin w = \sin^{-1} w$

If $z$ is real, $w$ is always between $-1$ and $+1$

If $z$ is complex, $w$ does not have to be between $-1$ and $+1$

It is convenient to use the forms,

$$w = \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$w = \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$
Example of $\cos z$, $\sin z$ with complex $z$

- Find $z = \arccos(i\sqrt{8})$ in the form $x + iy$

We start with $z = \arccos(i\sqrt{8})$ and write equivalently $\cos z = i\sqrt{8}$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = i\sqrt{8}$$

For simplicity take $u = e^{iz}$ then we can write,

$$\frac{u + u^{-1}}{2} = i\sqrt{8}$$

Which gives the quadratic equation,

$$u^2 - 4i\sqrt{2}u + 1 = 0$$

This has the roots $u = (2\sqrt{2} \pm 3)i$, so $iz = \ln[(2\sqrt{2} \pm 3)i]...$

complete in the homework!
For a pendulum oscillating with small angle $\theta$, or a mass-spring system obeying Hooke’s law, we get simple harmonic motion

\[
\frac{d^2 \theta}{dt^2} + \frac{g}{l} \theta = 0
\]

\[
\frac{d^2 y}{dt^2} + \frac{k}{m} y = 0
\]

$y(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t) = Ce^{i\omega_0 t}$. Note that $C$ is complex, but we usually take $A$ and $B$ to be real. Also $\omega_0 = \sqrt{\frac{k}{m}}$.

- $A$ and $B$ (or $C$) are determined by conditions at $t = 0$ (position and velocity)
For a pendulum oscillating with small angle $\theta$, or a mass-spring system obeying Hooke’s law, we get simple harmonic motion

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- $A$ and $B$ (or $C$) are determined by conditions at $t = 0$ (position and velocity)
- When using $y(t) = Ce^{i\omega_0 t}$, we assume $\text{Re}[y(t)]$ gives actual displacement
Simple-harmonic oscillator

For a pendulum oscillating with small angle $\theta$, or a mass-spring system obeying Hooke’s law, we get simple harmonic motion

$$\frac{d^2 \theta}{dt^2} + \frac{g}{l} \theta = 0$$

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- $A$ and $B$ (or $C$) are determined by conditions at $t = 0$ (position and velocity)
- When using $y(t) = Ce^{i\omega_0 t}$, we assume $\text{Re}[y(t)]$ gives actual displacement
- The $A$ and $B$ can be directly related to $C$ (homework)
Damped, driven simple-harmonic oscillator

Complex exponentials useful in the damped, driven oscillator

\[ \frac{d^2 y}{dt^2} + 2b \frac{dy}{dt} + \omega_0^2 y = F_D \cos(\omega t) \]

We will assume

\[ Y(t) = Ce^{i\omega t} = Ae^{i(\omega t + \phi)} \]

with the driving force \( F_D e^{i\omega t} \). The \( A \) and \( \delta \) are real and \( C \) is complex. Substitute and solve for \( C \),

\[ C = \frac{F_D}{\omega_0^2 - \omega^2 + 2ib\omega} \]

We would like to get \( C \) into the form \( Ae^{i\phi} \), so we multiply numerator and denominator by complex conjugate of denominator
Damped, driven simple-harmonic oscillator, cont.

\[ C = \frac{F_D}{\omega_0^2 - \omega^2 + 2ib\omega} \left[ \frac{\omega_0^2 - \omega^2 - 2ib\omega}{\omega_0^2 - \omega^2 - 2ib\omega} \right] = \frac{F_D}{(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2} \]

We now need \( A = |C| = \sqrt{C^*C} \) and also an expression for \( \phi \). We find

\[ A = |C| = \frac{F_D}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2}} \]

and for \( \phi \),

\[ \tan \phi = \frac{2b\omega}{\omega_0^2 - \omega^2} \]
Then we get \( Y(t) = Ae^{i(\omega t - \phi)} \) with the \( A \) and \( \phi \) given on last slide. Then finally, for a driving force \( F_D \cos \omega t \), we take the real part of \( Y(t) \),

\[
y(t) = A \cos(\omega t - \phi)
\]

Using complex exponentials was somewhat simpler, because otherwise we would have both sin and cos terms to keep track of, and also solve for \( A \) and \( B \) (see previously).
LCR circuit completely analogous to damped, driven simple-harmonic oscillator

Consider an alternating current source with potential $V_D = V_0 \cos \omega t$

$$V_L + V_R + V_C = V_D$$

$$L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = V_0 \cos \omega t$$

$$Q(t) = Q_0 e^{i(\omega t - \phi)}$$

Solve as a homework problem, but given damped-driven oscillator solution, this should be easy!
Damped, driven simple-harmonic oscillator

Is that everything for this problem?

► No! We only considered *particular* solutions
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▶ Particular solution valid for long-time behavior
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- We also need *homogeneous* solutions
- Homogeneous solutions relevant for no driving force and behavior right after driving force turned on
- This is a subject for more study in Chapter 8!