Chapter 9 – Rotation and Rolling

I. Rotational variables

- Angular position, displacement, velocity, acceleration

II. Rotation with constant angular acceleration

III. Relation between linear and angular variables

- Position, speed, acceleration

IV. Kinetic energy of rotation

V. Rotational inertia

VI. Torque

VII. Newton's second law for rotation

VIII. Work and rotational kinetic energy

IX. Rolling motion

I. Rotational variables

Rigid body: body that can rotate with all its parts locked together and without shape changes.

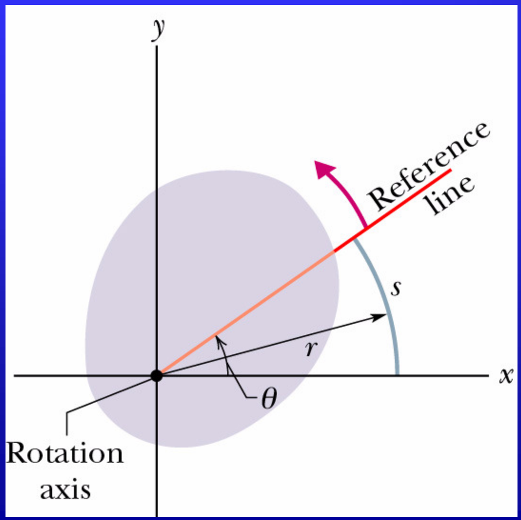
Rotation axis: every point of a body moves in a circle whose center lies on the rotation axis. Every point moves through the same angle during a particular time interval.

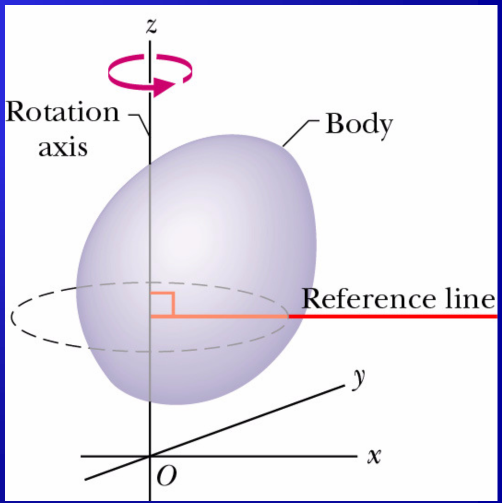
Reference line: fixed in the body, perpendicular to the rotation axis and rotating with the body.

Angular position: the angle of the reference line relative to the positive direction of the x-axis.

$$\theta = \frac{arc\ length}{radius} = \frac{s}{r}$$

Units: radians (rad)





$$1\ rev = 360^\circ = \frac{2\pi r}{r} = 2\pi\ rad$$

$$1\ rad = 57.3^\circ = 0.159\ rev$$

Note: we do not reset θ to zero with each complete rotation of the reference line about the rotation axis. 2 turns → $\theta = 4\pi$

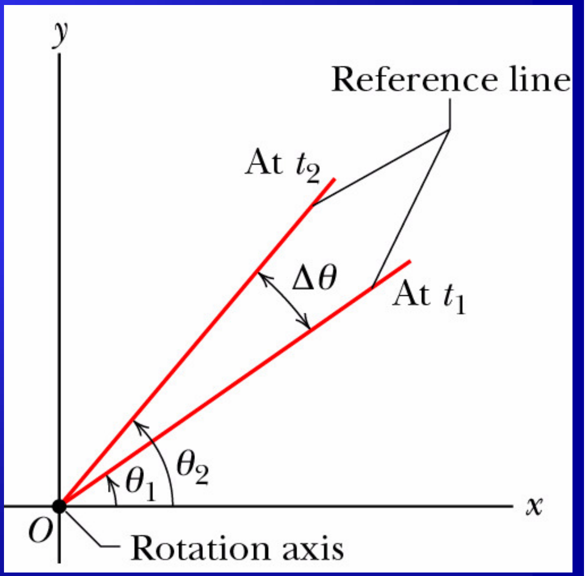
Translation: body's movement described by x(t).

Rotation: body's movement given by θ(t) = angular position of the body's reference line as function of time.

Angular displacement: body's rotation about its axis changing the angular position from θ_1 to θ_2.

$$\Delta\theta = \theta_2 - \theta_1$$

Clockwise rotation → negative
Counterclockwise rotation → positive



Angular velocity:

Average:

$$\omega_{avg} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

Instantaneous:

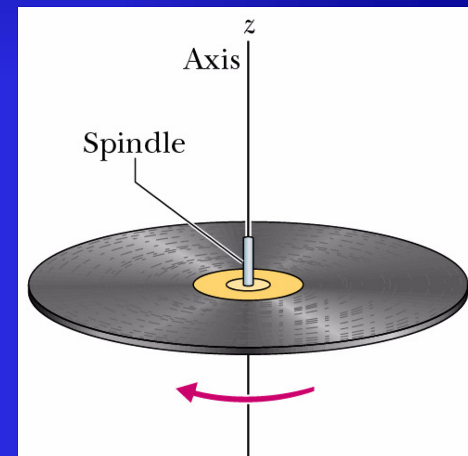
$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

Units: rad/s or rev/s

These equations hold not only for the rotating rigid body as a whole but also for every particle of that body because they are all locked together.

Angular speed (ω): magnitude of the angular velocity.

Angular acceleration:



Average:

$$\alpha_{avg} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

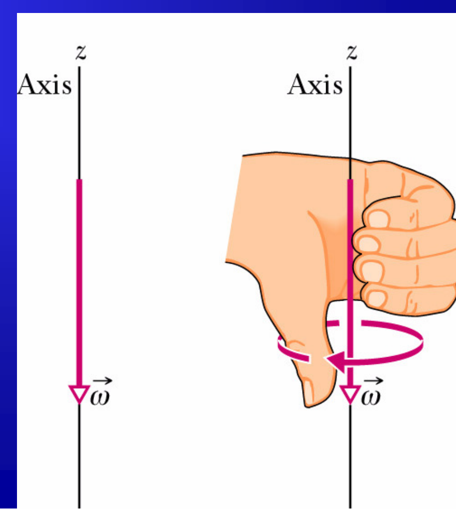
Instantaneous:

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

Angular quantities are "normally" vector quantities → right hand rule.

Examples: angular velocity, angular acceleration

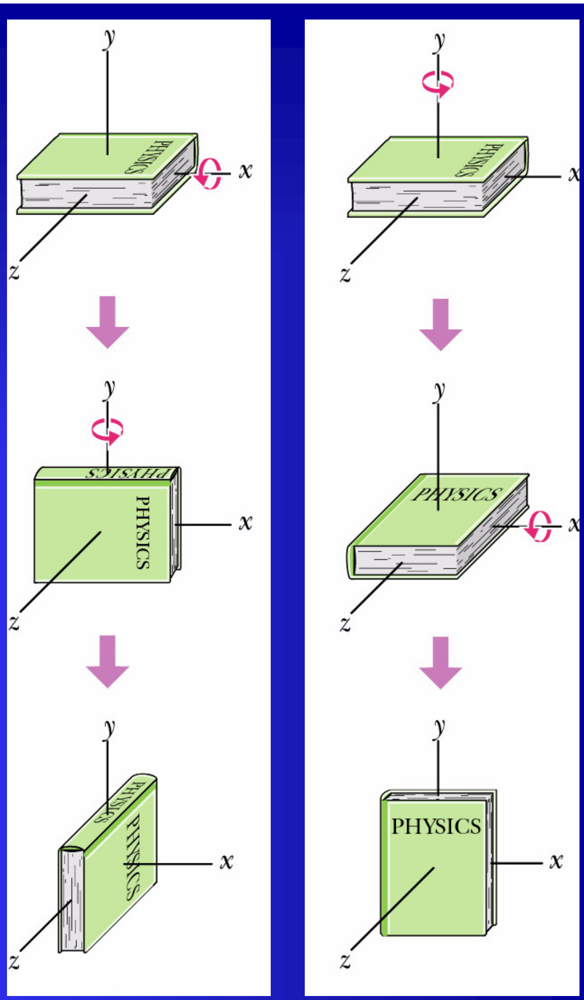
<u>Object rotates around the direction of the vector</u> → a vector defines an axis of rotation not the direction in which something is moving.



Angular quantities are "normally" vector quantities → right hand rule.

Exception: angular displacements

The order in which you add two angular displacements influences the final result → Δθ is not a vector.

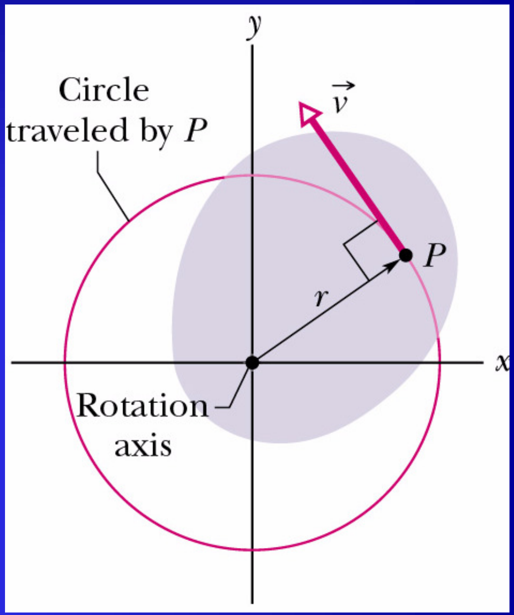


II. Rotation with constant angular acceleration

Linear equations	Angular equations
$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
$x - x_0 = v_0 t + \frac{1}{2} a t^2$	$\theta - \theta_0 = \omega_0 t + \frac{1}{2} \alpha t^2$
$v^2 = v_0^2 + 2a(x - x_0)$	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$
$x - x_0 = \frac{1}{2}(v_0 + v)t$	$\theta - \theta_0 = \frac{1}{2}(\omega_0 + \omega)t$
$x - x_0 = vt - \frac{1}{2} a t^2$	$\theta - \theta_0 = \omega t - \frac{1}{2} \alpha t^2$

III. Relation between linear and angular variables

Position: $s = \theta \cdot r$ θ always in radians



Speed: $\frac{ds}{dt} = r\frac{d\theta}{dt} \rightarrow v = \omega \cdot r$ ω in rad/s

$\vec{v}$ is tangent to the circle in which a point moves

Since all points within a rigid body have the same angular speed ω, points located at greater distance with respect to the rotational axis have greater linear (or tangential) speed, v.

If ω=constant, v=constant → each point within the body undergoes uniform circular motion.

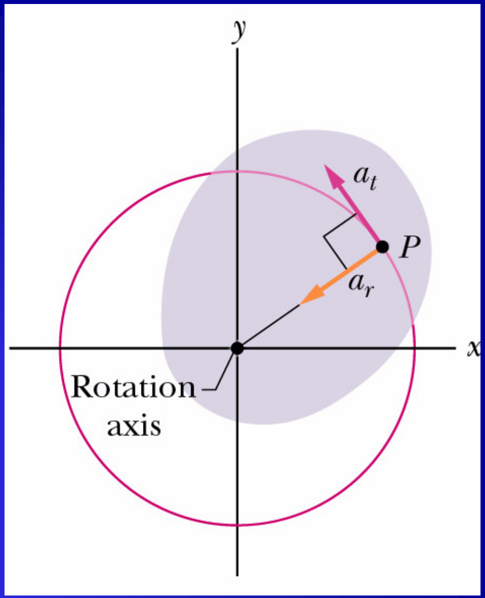
Period of revolution: $T = \frac{2\pi\ r}{v} = \frac{2\pi\ r}{\omega\ r} = \frac{2\pi}{\omega}$

Acceleration:

$$\frac{dv}{dt} = \frac{d(\omega \cdot r)}{dt} = \frac{d\omega}{dt} r = \alpha \cdot r \rightarrow \boxed{a_t = \alpha \cdot r}$$

Tangential component of linear acceleration

Responsible for changes in the magnitude of the linear velocity vector $\vec{v}$.



Radial component of linear acceleration:

$$a_r = \frac{v^2}{r} = \omega^2 \cdot r$$

Units: m/s^2

Responsible for changes in the direction of the linear velocity vector $\vec{v}$

IV. Kinetic energy of rotation

Reminder: **<u>Angular velocity</u>**, ω is the same for all particles within the rotating body.

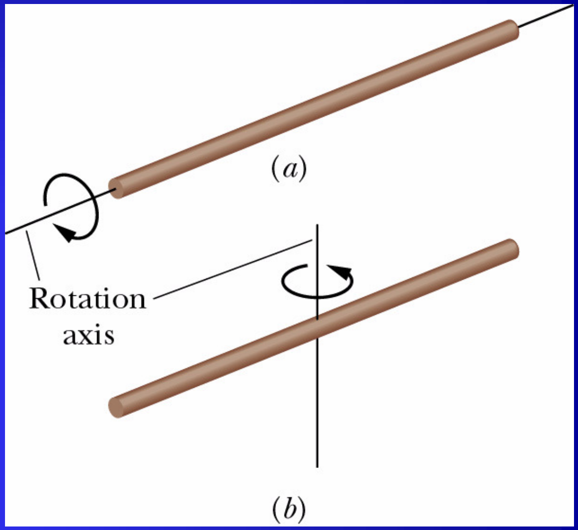
<u>Linear velocity</u>, v of a particle within the rigid body depends on the particle's distance to the rotation axis (r).

$$K = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 + \frac{1}{2}mv_3^2 + \ldots = \sum_i \frac{1}{2}m_i v_i^2 = \sum_i \frac{1}{2}m_i(\omega \cdot r_i)^2 = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2$$

Rotational inertia = Moment of inertia, I:

Indicates how the mass of the rotating body is distributed about its axis of rotation.

The moment of inertia is a <u>constant</u> for a particular rigid body and a <u>particular rotation axis</u>.



$$I = \sum_i m_i r_i^2$$

Example: long metal rod.

Units: kg m^2

Smaller rotational inertia in (a) → easier to rotate.

Kinetic energy of a body in pure rotation

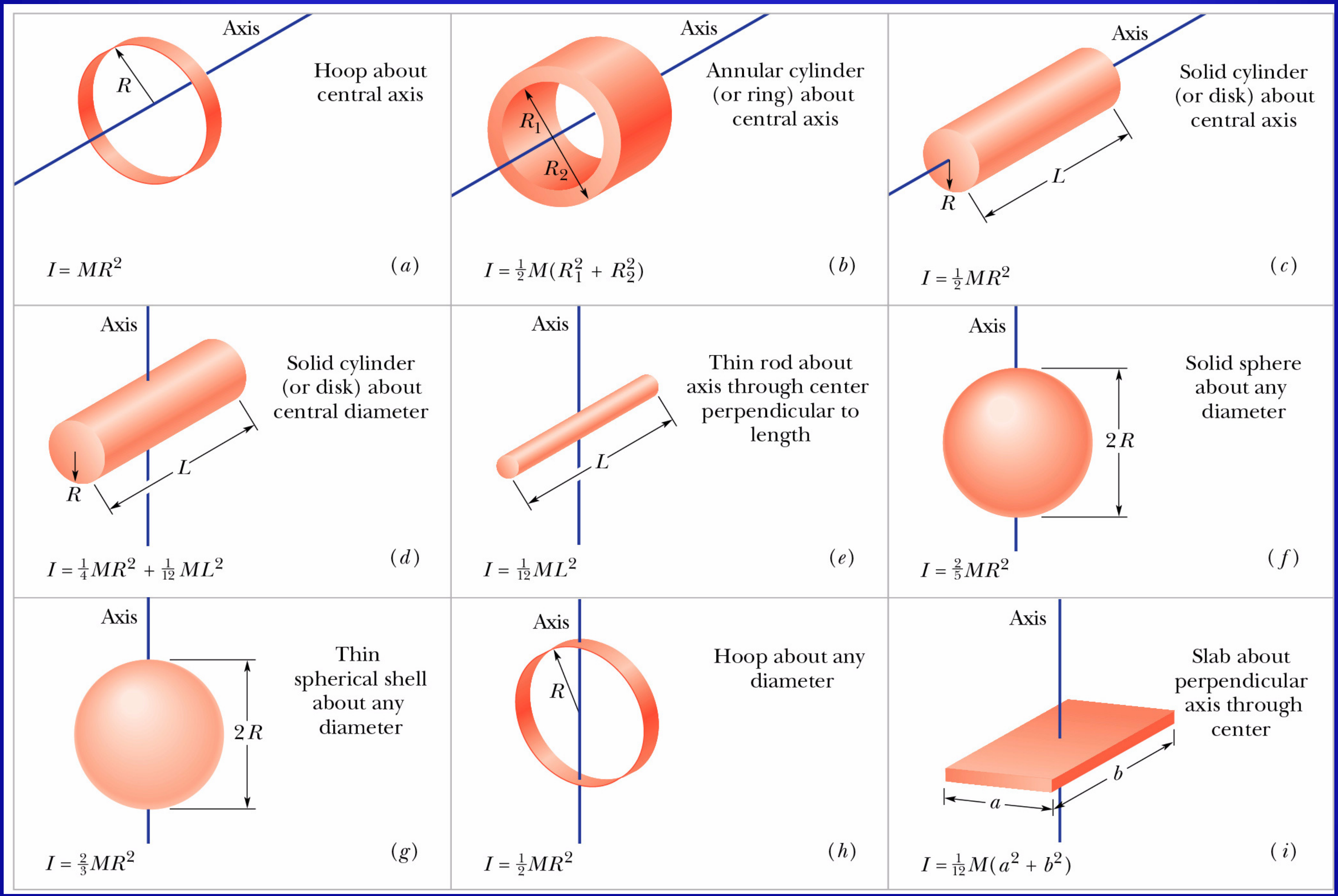
$$K = \frac{1}{2}I\omega^2$$

Kinetic energy of a body in pure translation

$$K = \frac{1}{2}Mv_{COM}^2$$

V. Rotational inertia

Discrete rigid body → $\boxed{I = \sum m_i r_i^2}$ Continuous rigid body → $\boxed{I = \int r^2\, dm}$

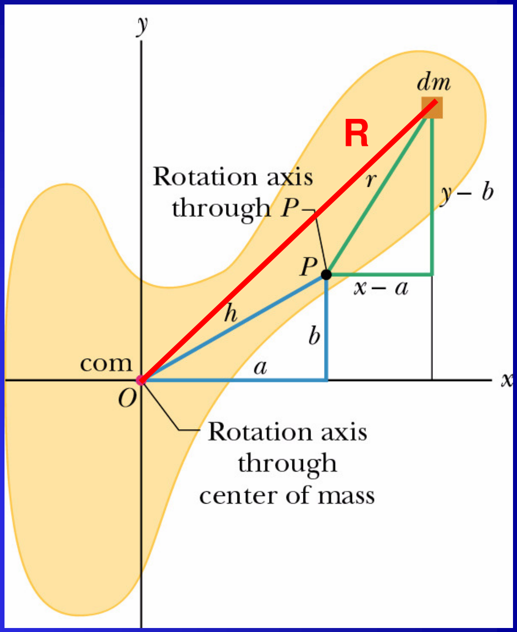


Parallel axis theorem

$$I = I_{COM} + Mh^2$$

h = perpendicular distance between the given axis and axis through COM.

Rotational inertia about a given axis = Rotational Inertia about a parallel axis that extends trough body's Center of Mass + Mh²



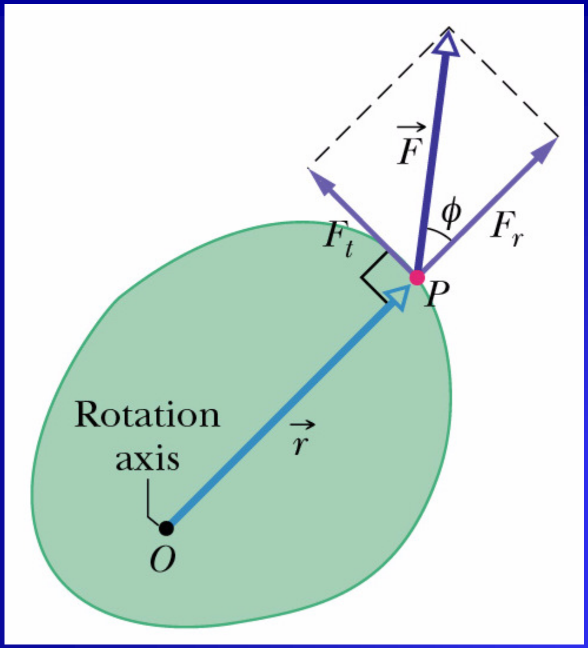
Proof:

$$I = \int r^2 dm = \int \left[(x-a)^2 + (y-b)^2 \right] dm = \int (x^2 + y^2) dm - 2a \int x dm - 2b \int y dm + \int (a^2 + b^2) dm$$

$$I = \int R^2 dm - 2a\cancel{M x_{COM}} - 2b\cancel{M y_{COM}} + Mh^2 = I_{COM} + Mh^2$$

VI. Torque

Torque: Twist → "Turning action of force $\vec{F}$ ".



Radial component, F_r : does not cause rotation → pulling a door parallel to door's plane.

Tangential component, F_t: does cause rotation → pulling a door perpendicular to its plane.

$F_t = F \sin\varphi$

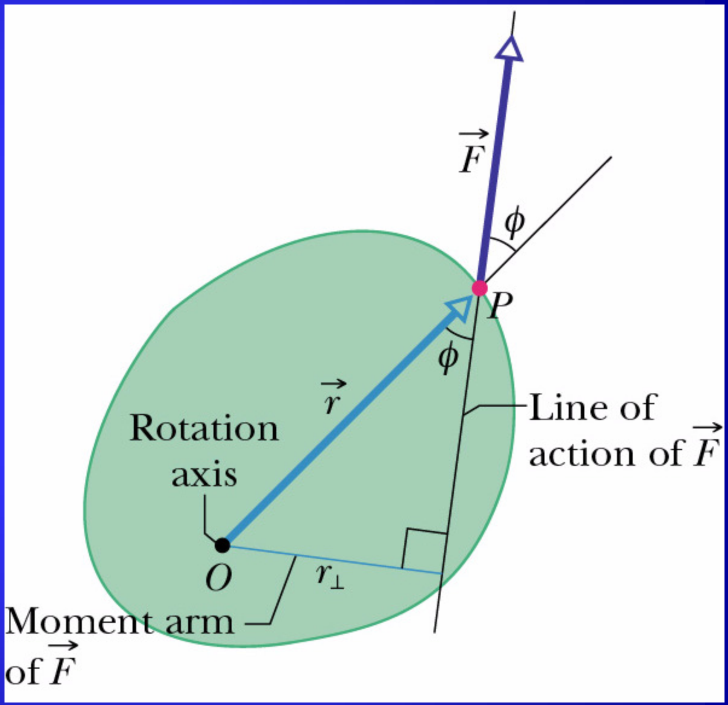
Units: Nm

$$\tau = r \cdot (F \cdot \sin\varphi) = r \cdot F_t = (r \sin\varphi) F = r_\perp F$$

$r_\perp$: Moment arm of $\vec{F}$

r : Moment arm of F_t

Vector quantity



Sign: Torque >0 if body rotates counterclockwise.
Torque <0 if clockwise rotation.

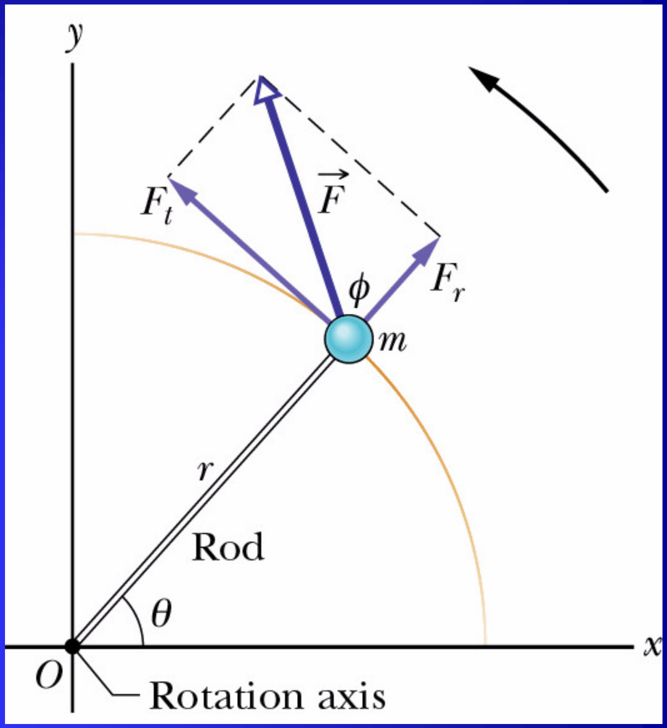
Superposition principle: When several torques act on a body, the net torque is the sum of the individual torques

VII. Newton's second law for rotation

$$F = ma \rightarrow \tau = I\alpha$$

Proof:

Particle can move only along the circular path → only the tangential component of the force F_t (tangent to the circular path) can accelerate the particle along the path.



$$F_t = ma_t$$

$$\tau = F_t \cdot r = ma_t \cdot r = m(\alpha \cdot r)r = (mr^2)\alpha = I\alpha$$

$$\tau_{net} = I\alpha$$

VIII. Work and Rotational kinetic energy

Translation	Rotation	
$\Delta K = K_f - K_i = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = W$	$\Delta K = K_f - K_i = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2 = W$	Work-kinetic energy Theorem
$W = \int_{x_i}^{x_f} F dx$	$W = \int_{\theta_i}^{\theta_f} \tau \cdot d\theta$	Work, rotation about fixed axis
$W = F \cdot d$	$W = \tau(\theta_f - \theta_i)$	Work, constant torque
$P = \frac{dW}{dt} = F \cdot v$	$P = \frac{dW}{dt} = \tau \cdot \omega$	Power, rotation about fixed axis

Proof:

$$W = \Delta K = K_f - K_i = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = \frac{1}{2}m(\omega_f r)^2 - \frac{1}{2}m(\omega_i r)^2 = \frac{1}{2}(mr^2)\omega_f^2 - \frac{1}{2}(mr^2)\omega_i^2 = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2$$

$$dW = F_t ds = F_t \cdot r \cdot d\theta = \tau \cdot d\theta \rightarrow W = \int_{\theta_i}^{\theta_f} \tau \cdot d\theta$$

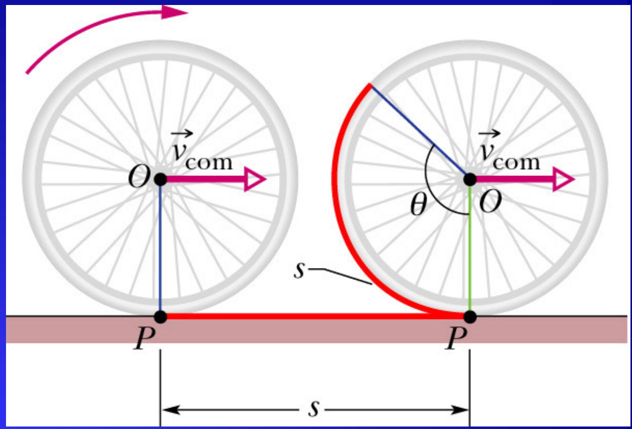
$$P = \frac{dW}{dt} = \frac{\tau \cdot d\theta}{dt} = \tau \cdot \omega$$

IX. Rolling

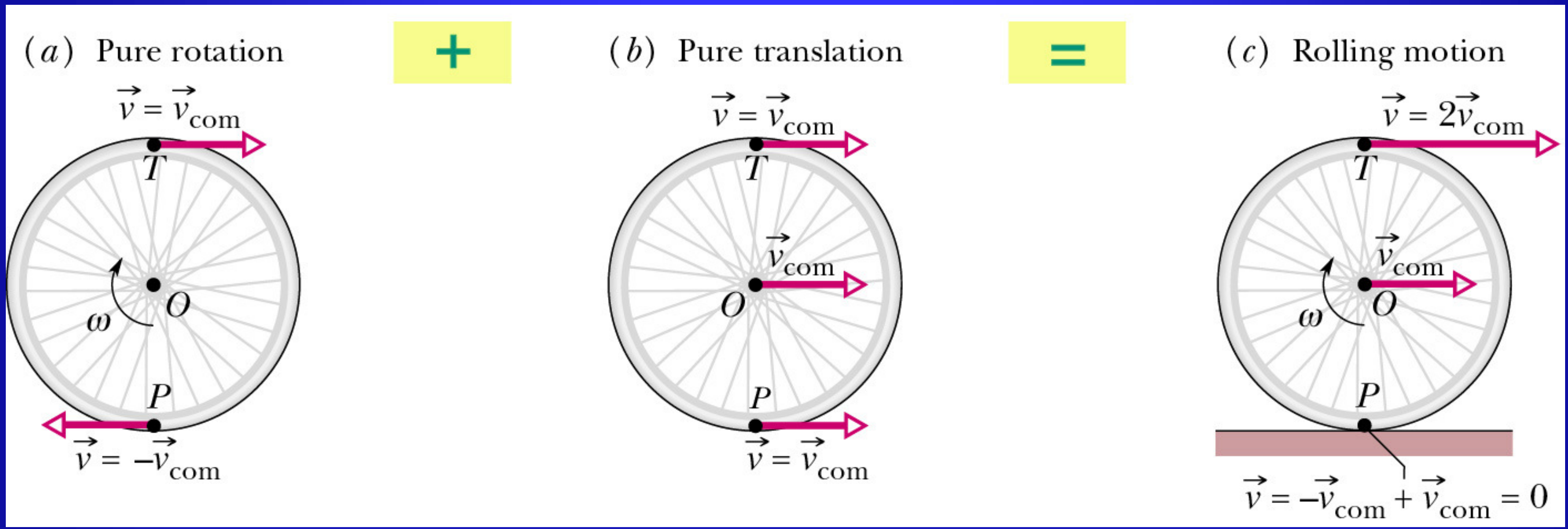
- Rotation + Translation combined.

Example: bicycle's wheel.

$$s = \theta \cdot R \rightarrow \frac{ds}{dt} = \frac{d\theta}{dt} R = \omega \cdot R = v_{COM}$$



Smooth rolling motion



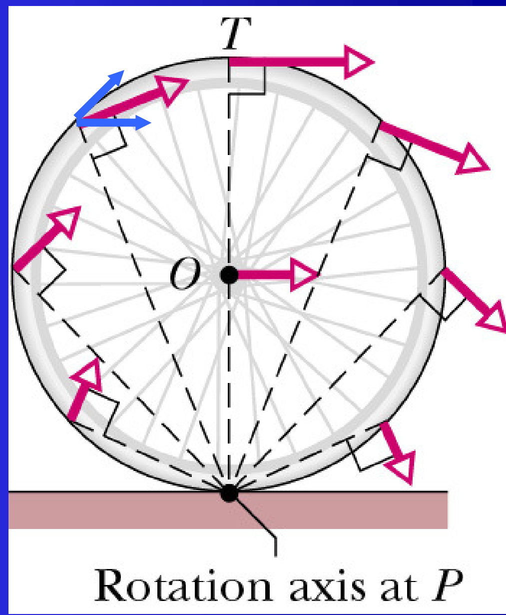
The motion of any round body rolling smoothly over a surface can be separated into purely rotational and purely translational motions.

- Pure rotation.

Rotation axis → through point where wheel contacts ground.

Angular speed about P = Angular speed about O for stationary observer.

$$v_{top} = (\omega)(2R) = 2(\omega R) = 2v_{COM}$$



Instantaneous velocity vectors = sum of translational and rotational motions.

- Kinetic energy of rolling.

$$I_p = I_{COM} + MR^2$$

$$K = \frac{1}{2} I_p \omega^2 = \frac{1}{2} I_{COM}\omega^2 + \frac{1}{2} MR^2\omega^2 = \frac{1}{2} I_{COM}\omega^2 + \frac{1}{2} Mv_{COM}^2$$

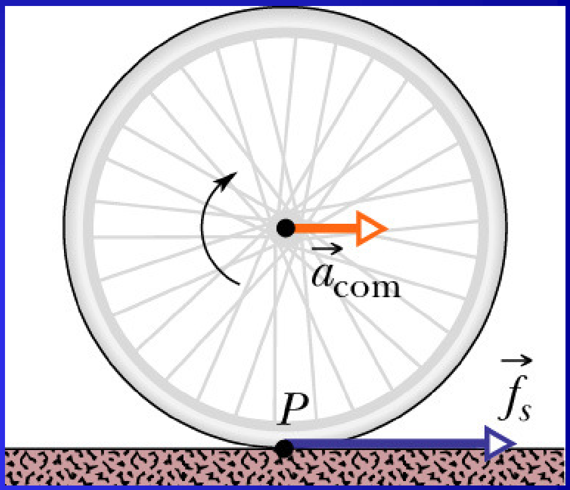
A rolling object has two types of kinetic energy → Rotational: 0.5 $I_{COM}\omega^2$ (about its COM).

Translational: 0.5 Mv^2_{COM} (translation of its COM).

- Forces of rolling.

(a) Rolling at constant speed → no sliding at P
→ no friction.

(b) Rolling with acceleration → sliding at P → friction force opposed to sliding.



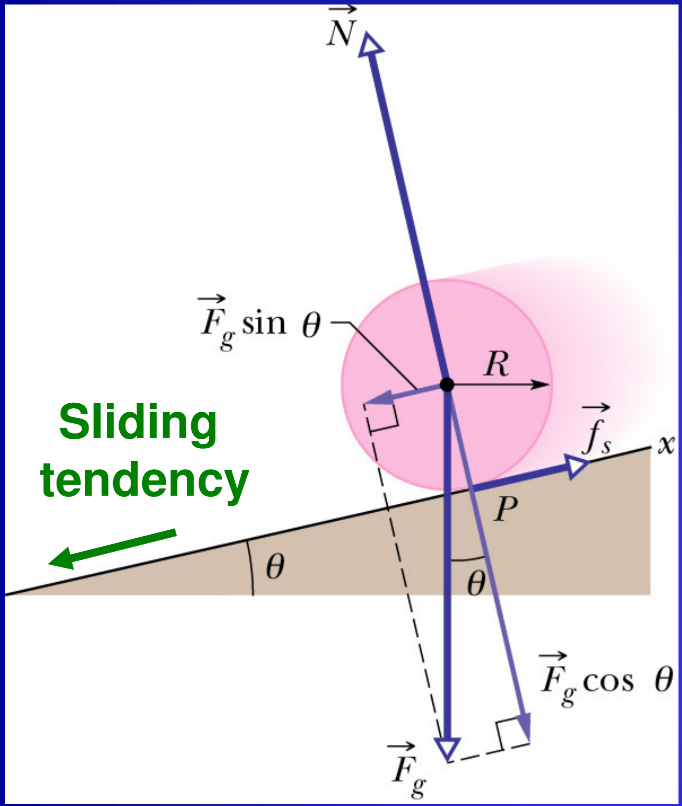
Sliding

Increasing acceleration

Static friction → wheel does not slide → **smooth rolling motion** → $\mathbf{a_{COM} = \alpha\, R}$

Example₁: wheels of a car moving forward while its tires are spinning madly, leaving behind black stripes on the road → rolling with slipping = skidding →Icy pavements.
Antiblock braking systems are designed to ensure that tires roll without slipping during braking.

Example$_2$: ball rolling smoothly down a ramp. (No slipping).



1. Frictional force causes the rotation. Without friction the ball will not roll down the ramp, will just slide.

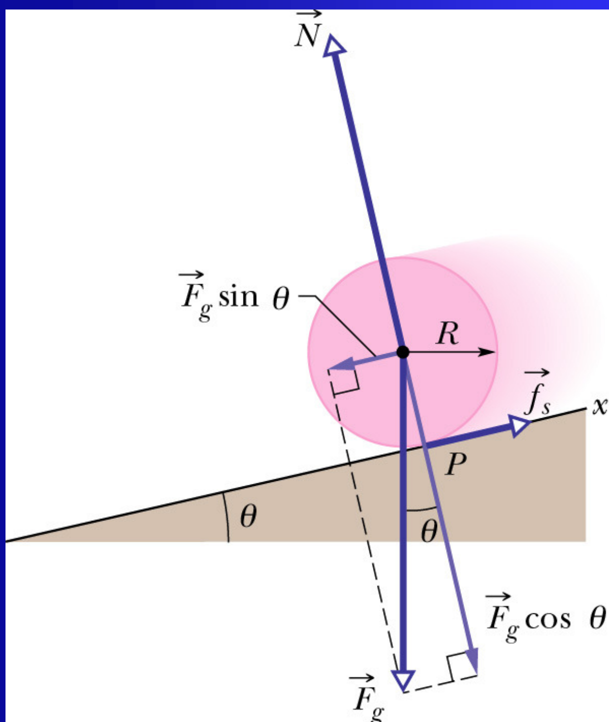
2. Rolling without sliding → the point of contact between the sphere and the surface is at rest → the frictional force is the static frictional force.

3. Work done by frictional force = 0 → the point of contact is at rest (static friction).

Example: ball rolling smoothly down a ramp.

$$F_{net,x} = ma_x \rightarrow f_s - Mg\sin\theta = Ma_{COM,x}$$

Note: Do not assume $f_s = f_{s,max}$. The only f_s requirement is that its magnitude is just right for the body to roll smoothly down the ramp, without sliding.



Newton's second law in angular form

→ Rotation about center of mass

$$\tau = r_\perp F \rightarrow \tau_{f_s} = R \cdot f_s$$
$$\tau_{F_g} = \tau_N = 0$$

$$\tau_{net} = I\alpha \rightarrow R \cdot f_s = I_{COM}\alpha = I_{COM}\frac{-a_{COM,x}}{R}$$

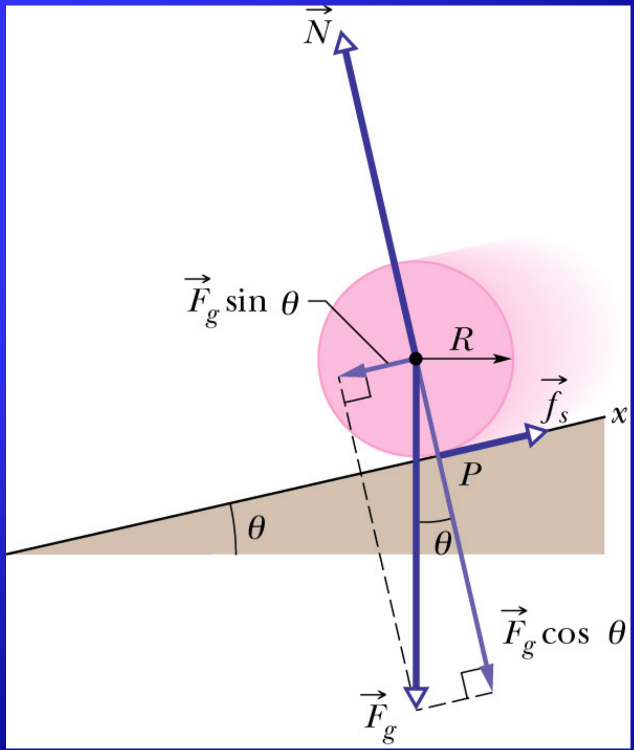
$$\rightarrow f_s = -I_{COM}\frac{a_{COM,x}}{R^2}$$

$$f_s - Mg\sin\theta = Ma_{COM,x}$$

$$f_s = -I_{COM}\frac{a_{COM,x}}{R^2} = Mg\sin\theta + Ma_{COM,x} \rightarrow -(M + \frac{I_{COM}}{R^2})a_{COM,x} = Mg\sin\theta$$

$$\boxed{a_{COM,x} = -\frac{g\sin\theta}{1 + I_{com}/MR^2}}$$

Linear acceleration of a body rolling along an incline plane



Example: ball rolling smoothly down a ramp of height h

Conservation of Energy

$$K_f + U_f = K_i + U_i$$

$$0.5 I_{COM}\omega^2 + 0.5 M v_{COM}^2 + 0 = 0 + Mgh$$

$$0.5 I_{COM}\frac{v_{COM}^2}{R^2} + 0.5 M v_{COM}^2 + 0 = 0 + Mgh$$

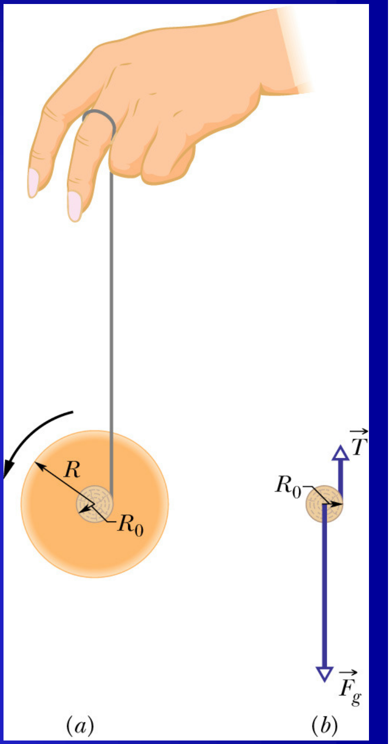
$$0.5 v_{COM}^2\left(\frac{I_{COM}}{R^2} + M\right) = Mgh$$

$$v_{COM} = \left(\frac{2hg}{1+\left(\frac{I_{COM}}{MR^2}\right)}\right)^{1/2}$$

Although there is friction (static), there is no loss of Emec because the point of contact with the surface is at rest relative to the surface at any instant

- Yo-yo

Potential energy (mgh)→ kinetic energy: translational ($0.5mv^2_{COM}$) and rotational ($0.5\ I_{COM}\omega^2$)



Analogous to body rolling down a ramp:

- Yo-yo rolls down a string at an angle θ =90º with the horizontal.
- Yo-yo rolls on an axle of radius R_0.
- Yo-yo is slowed by the tension on it from the string.

$$a_{COM,x} = \frac{-g\sin\theta}{1 + I_{com}/MR^2} = \frac{-g}{1 + I_{com}/MR_0^{\ 2}}$$