Chapter 11 – Torque and Angular Momentum

I. Torque

II. Angular momentum

- Definition

III. Newton's second law in angular form

IV. Angular momentum

- System of particles
- Rigid body
- Conservation

I. Torque

- Vector quantity.

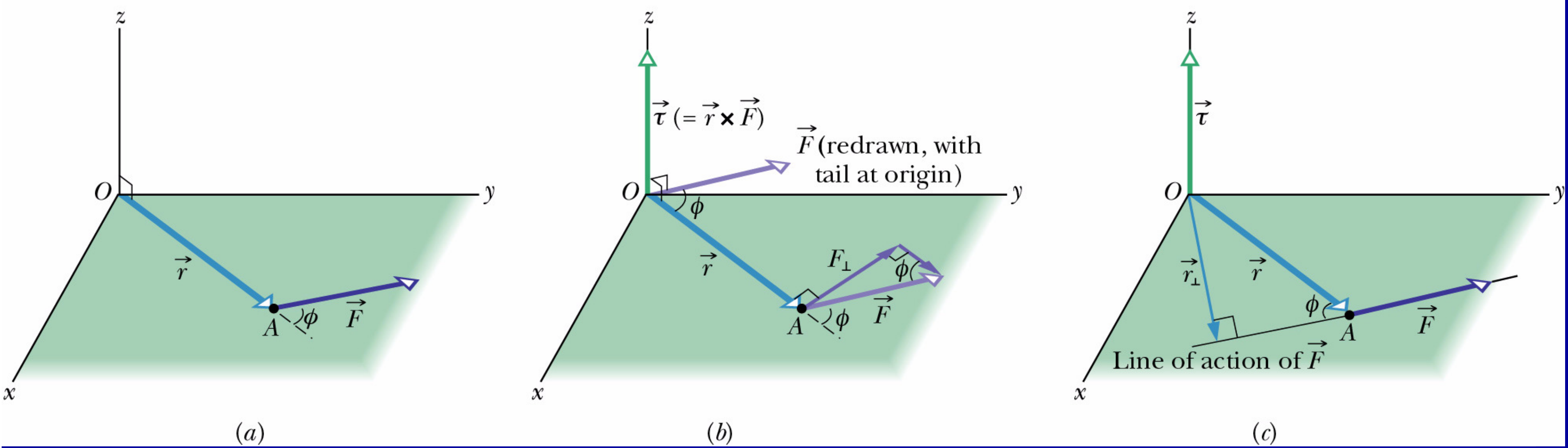
$$\vec{\tau} = \vec{r} \times \vec{F}$$

Direction: right hand rule.

Magnitude:

$$\tau = r \cdot F \sin\varphi = r \cdot F_{\perp} = (r \sin\varphi) F = r_{\perp} F$$

Torque is calculated with respect to (about) a point. Changing the point can change the torque's magnitude and direction.



II. Angular momentum

- Vector quantity. $\vec{l} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$ Units: kg m²/s

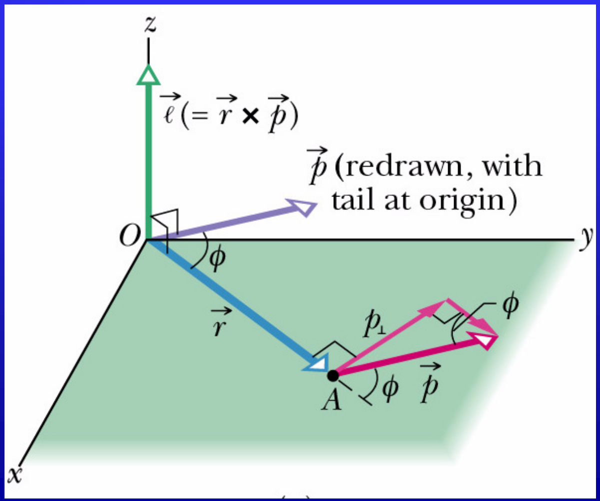
Magnitude: $l = r \cdot p \sin\varphi = r \cdot m \cdot v \sin\varphi = r \cdot m \cdot v_\perp = r \cdot p_\perp = (r \sin\varphi) p = r_\perp p = r_\perp m \cdot v$

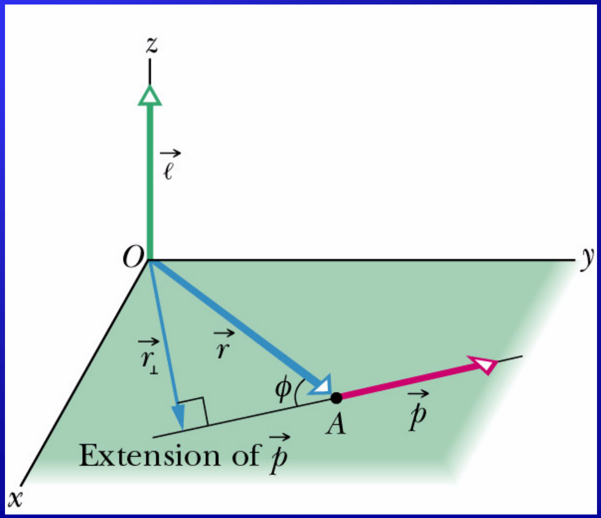
Direction: right hand rule.

$\vec{l}$ positive → counterclockwise

$\vec{l}$ negative → clockwise

Direction of $\vec{l}$ is always perpendicular to plane formed by $\vec{r}$ and $\vec{p}$.





III. Newton's second law in angular form

Linear

$$\vec{F}_{net} = \frac{d\vec{p}}{dt}$$

Angular

$$\vec{\tau}_{net} = \frac{d\vec{l}}{dt}$$

Single particle

The vector sum of all torques acting on a particle is equal to the time rate of change of the angular momentum of that particle.

Proof:

$$\vec{l} = m(\vec{r}\times\vec{v}) \rightarrow \frac{d\vec{l}}{dt} = m\left(\vec{r}\times\frac{d\vec{v}}{dt} + \frac{d\vec{r}}{dt}\times\vec{v}\right) = m(\vec{r}\times\vec{a} + \vec{v}\times\vec{v}) = m(\vec{r}\times\vec{a}) =$$

$$\frac{d\vec{l}}{dt} = \vec{r}\times m\vec{a} = \vec{r}\times\vec{F}_{net} = \sum\left(\vec{r}\times\vec{F}\right) = \vec{\tau}_{net}$$

V. Angular momentum

- System of particles:

$$L = \vec{l}_1 + \vec{l}_2 + \vec{l}_3 + \ldots + \vec{l}_n = \sum_{i=1}^{n} \vec{l}_i$$

$$\frac{d\vec{L}}{dt} = \sum_{i=1}^{n} \frac{d\vec{l}_i}{dt} = \sum_{i=1}^{n} \vec{\tau}_{net,i} \rightarrow \vec{\tau}_{net} = \frac{d\vec{L}}{dt}$$

Includes internal torques (due to forces between particles within system) and external torques (due to forces on the particles from bodies outside system).

Forces inside system → third law force pairs → torque_{int} sum =0 → The only torques that can change the angular momentum of a system are the external torques acting on a system.

The net external torque acting on a system of particles is equal to the time rate of change of the system's total angular momentum $\vec{L}$.

- Rigid body (rotating about a fixed axis with constant angular speed ω):

Magnitude $$l_i = (r_i)(p_i)(\sin 90^\circ) = (r_i)(m_i v_i)$$

$$v_i = \omega \cdot r_i$$

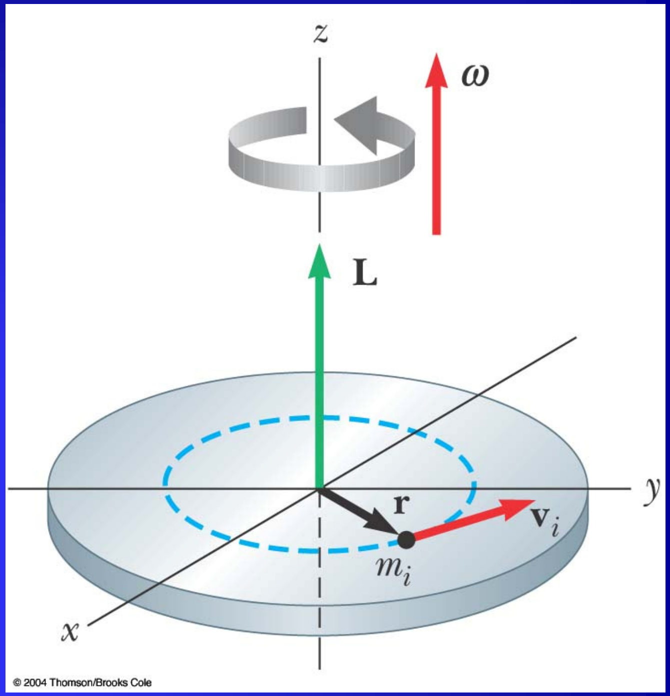
$$l_i = r_i m_i (\omega\, r_i) = \omega\, m_i r_i^2$$

Direction: $\vec{l}_i$ → perpendicular to $\vec{r}_i$ and $\vec{p}_i$

$$L_z = \sum_{i=1}^{n} l_{iz} = \sum_{i=1}^{n} m_i r_i^2 \omega = \left(\sum_{i=1}^{n} m_i \cdot r_i^2 \right) \omega = I\omega$$

$$L_z = \omega\, I$$

$$\frac{dL_z}{dt} = I \frac{d\omega}{dt} = I\alpha \rightarrow \frac{dL_z}{dt} = \tau_{ext}$$



$$L = I\omega$$

Rotational inertia of a rigid body about a fixed axis

- Conservation of angular momentum:

Newton's second law

$$\vec{\tau}_{net} = \frac{d\vec{L}}{dt}$$

If no net external torque acts on the system → (isolated system)

$$\frac{d\vec{L}}{dt} = 0 \rightarrow \vec{L} = cte$$

Law of conservation of angular momentum:

$$\vec{L}_i = \vec{L}_f \qquad (isolated\ system)$$

Net angular momentum at time t_i = Net angular momentum at later time t_f

If the net external torque acting on a system is zero, the angular momentum of the system remains constant, no matter what changes take place within the system.

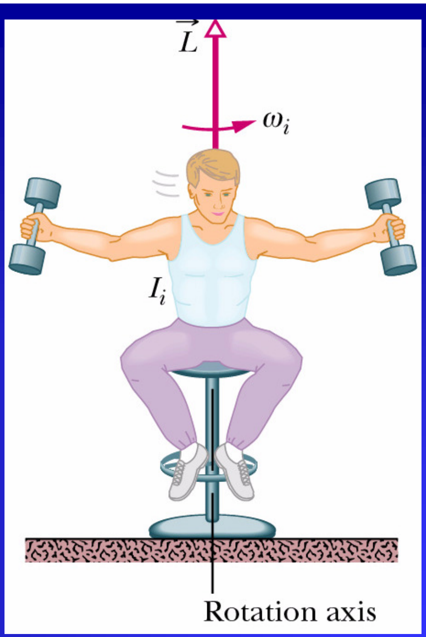
If the component of the net external torque on a system along a certain axis is zero, the component of the angular momentum of the system along that axis cannot change, no matter what changes take place within the system.

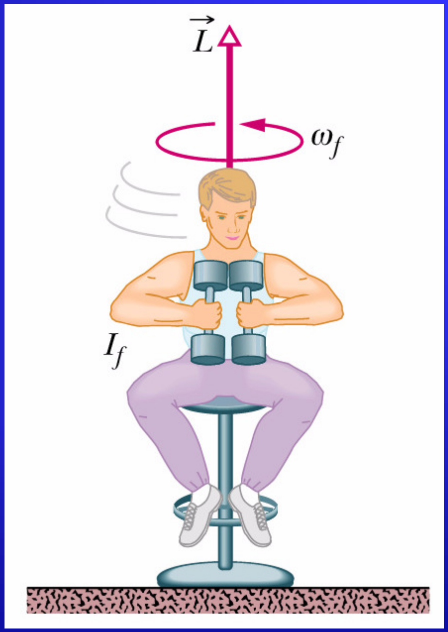
This conservation law holds not only within the frame of Newton's mechanics but also for relativistic particles (speeds close to light) and subatomic particles.

$$I_i\omega_i = I_f\omega_f$$

($I_{i,f}$, $\omega_{i,f}$ refer to rotational inertia and angular speed before and after the redistribution of mass about the rotational axis).

Examples:



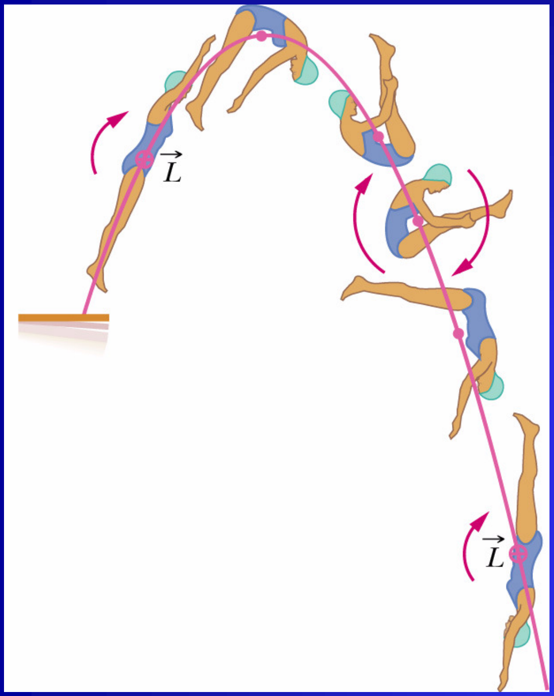


Spinning volunteer

$I_f < I_i$ (mass closer to rotation axis)

Torque ext =0 → $I_i\omega_i = I_f\,\omega_f$

$\omega_f > \omega_i$



Springboard diver

- Center of mass follows parabolic path.

- When in air, no net external torque about COM → Diver's angular momentum $\vec{L}$ constant throughout dive (magnitude and direction).

- $\vec{L}$ is perpendicular to the plane of the figure (inward).

- Beginning of dive → She pulls arms/legs closer

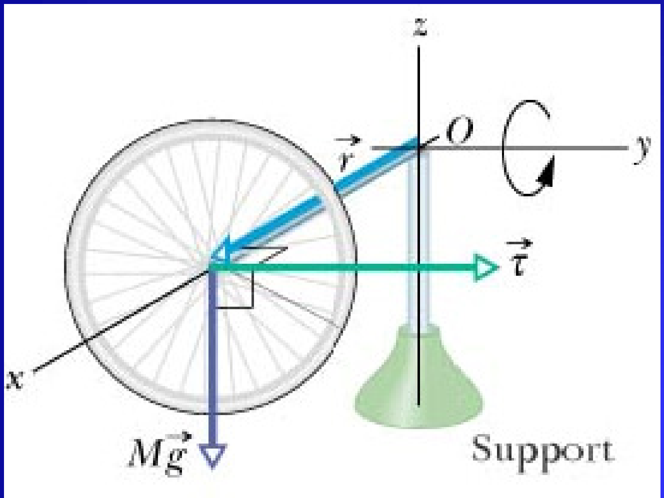
Intention: I is reduced → ω increases

- End of dive → layout position

Purpose: I increases → slow rotation rate → less "water-splash"

Translation		Rotation	
Force	$\vec{F}$	**Torque**	$\vec{\tau} = \vec{r} \times \vec{F}$
Linear momentum	$\vec{p}$	**Angular momentum**	$\vec{l} = \vec{r} \times \vec{p}$
Linear momentum (system of particles, rigid body)	$\vec{P} = \sum_i \vec{p}_i = M\vec{v}_{COM}$	**Angular momentum**	$\vec{L} = \sum_i \vec{l}_i$ System of particles $L = I\omega$ Rigid body, fixed axis L=component along that axis.
Newton's second law	$\vec{F} = \frac{d\vec{P}}{dt}$	**Newton's second law**	$\vec{\tau}_{net} = \frac{d\vec{L}}{dt}$
Conservation law (Closed isolated system)	$\vec{P} = cte$	**Conservation law** (Closed isolated system)	$\vec{L} = cte$

IV. Precession of a gyroscope



Gyroscope: wheel fixed to shaft and free to spin about shaft's axis.

Non-spinning gyroscope

If one end of shaft is placed on a support and released → Gyroscope falls by rotating downward about the tip of the support.

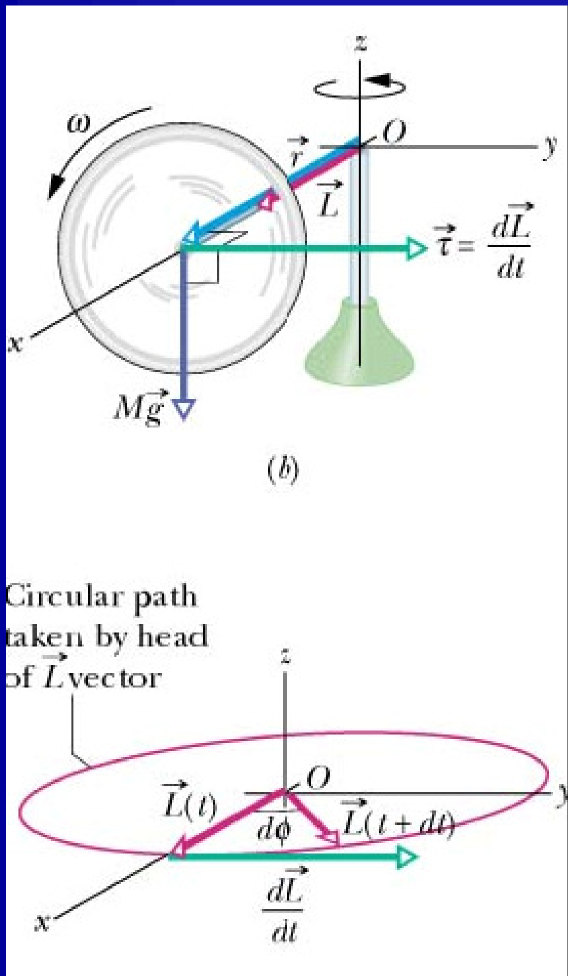
$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

The torque causing the downward rotation (fall) changes angular momentum of gyroscope.

Torque → caused by gravitational force acting on COM.

$$\tau = Mgr\sin 90^\circ = Mgr$$

Rapidly spinning gyroscope



If released with shaft's angle slightly upward → first rotates downward, then spins horizontally about vertical axis z → precession due to non-zero initial angular momentum

Simplification: i) L due to rapid spin >> L due to precession
ii) shaft horizontal when precession starts

$$L = I\omega$$

I = rotational moment of gyroscope about shaft
ω = angular speed of wheel about shaft

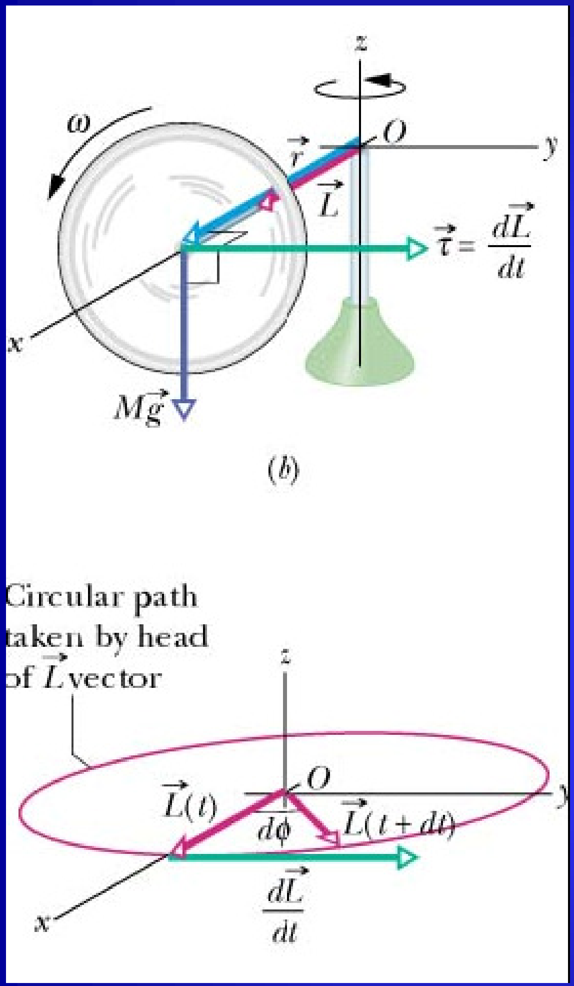
Vector $\vec{L}$ along shaft, parallel to $\vec{r}$

Torque perpendicular to $\vec{L}$ → can only change the Direction of L, not its magnitude.

$$d\vec{L} = \vec{\tau}dt \rightarrow dL = \tau dt = Mgrdt$$

$$d\varphi = \frac{dL}{L} = \frac{Mgrdt}{I\omega}$$

Rapidly spinning gyroscope



$$d\vec{L} = \vec{\tau}dt \rightarrow dL = \tau dt = Mgrdt$$

$$d\varphi = \frac{dL}{L} = \frac{Mgrdt}{I\omega}$$

Precession rate:

$$\Omega = \frac{d\varphi}{dt} = \frac{Mgr}{I\omega}$$