Chapter 30 – Inductance

- Mutual Inductance
- Self-Inductance and Inductors
- Magnetic-Field Energy
- The R-L Circuit
- The L-C Circuit
- The L-R-C Series Circuit
1. **Mutual Inductance**

- A changing current in coil 1 causes B and a changing magnetic flux through coil 2 that induces emf in coil 2.

\[ \mathcal{E}_2 = -N_2 \frac{d\Phi_{B2}}{dt} \]

Magnetic flux through coil 2:

\[ N_2 \Phi_{B2} = M_{21}i_1 \]

Mutual inductance of two coils: \( M_{21} \)

\[ N_2 \frac{d\Phi_{B2}}{dt} = M_{21} \frac{di_1}{dt} \]

\[ \mathcal{E}_2 = -M_{21} \frac{di_1}{dt} \quad \rightarrow \quad M_{21} = \frac{N_2 \Phi_{B2}}{i_1} \]

\( M_{21} \) is a constant that depends on geometry of the coils. \( = M_{12} \).
- If a magnetic material is present, \( M_{21} \) will depend on magnetic properties. If relative permeability \( (K_m) \) is not constant \( (M \) not proportional to \( B) \) \( \rightarrow \) \( \Phi_{B_2} \) not proportional to \( i_1 \) (exception).

\[
\begin{align*}
\varepsilon_2 &= -M \frac{di_1}{dt} \\
\varepsilon_1 &= -M \frac{di_2}{dt}
\end{align*}
\]

emf opposes the flux change

Mutual inductance:

\[
M = \frac{N_2 \Phi_{B_2}}{i_1} = \frac{N_1 \Phi_{B_1}}{i_2}
\]

- Only a time-varying current induces an emf.

Units of inductance: 1 Henry = 1 Weber/A = 1 V s/A = 1 J/A\(^2\)

Ex. 30.1

Cross-sectional area \( A \)

Blue coil: \( N_2 \) turns

Black coil: \( N_1 \) turns
2. Self Inductance and Inductors

- When a current is present in a circuit, it sets up B that causes a magnetic flux that changes when the current changes → emf is induced.

**Lenz’s law**: a self-induced emf opposes the change in current that caused it → Induced emf makes difficult variations in current.

\[
L = \frac{N\Phi_B}{i}
\]

**Self-inductance**

\[
N \frac{d\Phi_B}{dt} = L \frac{di}{dt}
\]

**Self-induced emf**

\[
\varepsilon = -L \frac{di}{dt}
\]

**Self-inductance**: If the current \(i\) in the coil is changing, the changing flux through the coil induces an emf in the coil.
Inductors as Circuit Elements

Inductors oppose variations in the current through a circuit.

- In DC-circuit, L helps to maintain a steady current (despite fluctuations in applied emf). In AC circuit, L helps to suppress fast variations in current.

- Reminder of Kirchhoff’s loop rule: the sum of potential differences around any closed loop is zero because E produced by charges distributed around circuit is conservative $E_c$.

- The magnetically induced electric field within the coils of an inductor is non-conservative ($E_n$).

- If $R = 0$ in inductor’s coils $\rightarrow$ very small $E$ required to move charge through coils $\rightarrow$ total $E$ through coils $E_c + E_n = 0$. Since $E_c \neq 0$ $\rightarrow$ $E_c = -E_n$. Accumulation of charge on inductor’s terminals and surfaces of its conductors to produce that field.

$$\int E_n dl = -L \frac{di}{dt}$$
- $E_n \neq 0$ only within the inductor.

\[
\int_{a}^{b} \vec{E}_n \cdot d\vec{l} = -L \frac{di}{dt} \quad \vec{E}_c + \vec{E}_n = 0 \quad \text{(at each point within the inductor's coil)}
\]

\[
\int_{a}^{b} \vec{E}_c \cdot d\vec{l} = L \frac{di}{dt}
\]

- Self-induced emf opposes changes in current.

Potential difference between terminals of an inductor:

\[
V_{ab} = V_a - V_b = L \frac{di}{dt}
\]

$V_{ab}$ is associated with conservative, electrostatic forces, despite the fact that $E$ associated with the magnetic induction is non-conservative $\rightarrow$ Kirchhoff's loop rule can be used.

- If magnetic flux is concentrated in region with a magnetic material $\rightarrow \mu_0$ in eqs. must be replaced by $\mu = K_m \mu_0$. 
(a) Resistor with current $i$ flowing from $a$ to $b$: potential drops from $a$ to $b$.

\[ V_{ab} = iR > 0 \]

(b) Inductor with constant current $i$ flowing from $a$ to $b$: no potential difference.

\[ i \text{ constant: } \frac{di}{dt} = 0 \]

\[ V_{ab} = L \frac{di}{dt} = 0 \]

(c) Inductor with increasing current $i$ flowing from $a$ to $b$: potential drops from $a$ to $b$.

\[ i \text{ increasing: } \frac{di}{dt} > 0 \]

\[ V_{ab} = L \frac{di}{dt} > 0 \]

(d) Inductor with decreasing current $i$ flowing from $a$ to $b$: potential increases from $a$ to $b$.

\[ i \text{ decreasing: } \frac{di}{dt} < 0 \]

\[ V_{ab} = L \frac{di}{dt} < 0 \]
3. **Magnetic-Field Energy**

- Establishing a current in an inductor requires an input of energy. An inductor carrying a current has energy stored in it.

**Energy Stored in an Inductor**

Rate of transfer of energy into L: \[ P = V_{ab}i = L \cdot i \cdot \frac{di}{dt} \]

Energy supplied to inductor during \( dt \): \[ dU = P \, dt = L \, i \, di \]

Total energy \( U \) supplied while the current increases from zero to \( I \):

\[
U = L \int_{0}^{I} i \cdot di = \frac{1}{2} LI^2
\]

- Energy flows into an ideal (\( R = 0 \)) inductor when current in inductor increases. The energy is not dissipated, but stored in \( L \) and released when current decreases.
Magnetic Energy Density

-The energy in an inductor is stored in the magnetic field within the coil, just as the energy of a capacitor is stored in the electric field between its plates.

Ex: toroidal solenoid (B confined to a finite region of space within its core).

\[ V = (2\pi r) \, A \]

\[ L = \frac{\mu_0 N^2 A}{2\pi \cdot r} \]

\[ U = \frac{1}{2} LI^2 = \frac{1}{2} \frac{\mu_0 N^2 A}{2\pi \cdot r} I^2 \]

Energy per unit volume: \[ u = \frac{U}{V} \]
magnetic energy density

\[ u = \frac{U}{V} = \frac{U}{2\pi \cdot r \cdot A} = \frac{1}{2} \frac{\mu_0 N^2 I^2}{(2\pi \cdot r)^2} \]
\[ B = \frac{\mu_0 N I}{2\pi \cdot r} \]

\[ \frac{N^2 I^2}{(2\pi \cdot r)^2} = \frac{B^2}{\mu_0^2} \]

\[
\begin{array}{ll}
\text{Magnetic energy density in vacuum} & \Rightarrow \\
\frac{B^2}{2\mu_0} & \\
\text{Magnetic energy density in a material} & \Rightarrow \\
\frac{B^2}{2\mu} & \\
\end{array}
\]
4. The R-L Circuit

- An inductor in a circuit makes it difficult for rapid changes in current to occur due to induced emf.

Current-Growth in an R-L Circuit

At \( t = 0 \rightarrow \) Switch 1 closed.

\[
\begin{align*}
    v_{ab} &= I \cdot R \\
    v_{bc} &= L \frac{di}{dt} \\
    \varepsilon - i \cdot R - L \frac{di}{dt} &= 0 \\
    \frac{di}{dt} &= \frac{\varepsilon - iR}{L} = \frac{\varepsilon}{L} - \frac{R}{L} i
\end{align*}
\]

Closing switch \( S_1 \) connects the \( R-L \) combination in series with a source of emf \( \varepsilon \).

Closing switch \( S_2 \) while opening switch \( S_1 \) disconnects the combination from the source.
\[
\left(\frac{di}{dt}\right)_{\text{initial}} = \frac{\varepsilon}{L} \quad (t = 0 \Rightarrow i = 0 \Rightarrow V_{ab} = 0)
\]

\[
\left(\frac{di}{dt}\right)_{\text{final}} = 0 = \frac{\varepsilon}{L} - \frac{R}{L} I \quad I = \frac{\varepsilon}{R} \quad (t_f \Rightarrow \text{di/dt} = 0)
\]

\[
\frac{di}{i - \left(\frac{\varepsilon}{R}\right)} = -\frac{R}{L} dt
\]

\[
\int_{0}^{i} \frac{di'}{i' - \left(\frac{\varepsilon}{R}\right)} = -\int_{0}^{t} \frac{R}{L} dt'
\]

\[
\ln\left(\frac{i - \left(\frac{\varepsilon}{R}\right)}{-\varepsilon/R}\right) = -\frac{R}{L} t
\]

Current in R-L circuit:

\[
i = \frac{\varepsilon}{R} \left(1 - e^{-\left(\frac{R}{L}\right)t}\right)
\]
\[ \frac{di}{dt} = \frac{\epsilon}{L} e^{-(R/L)t} \]

\( t = 0 \rightarrow i = 0, \quad \frac{di}{dt} = \frac{\epsilon}{L} \)

\( t = \infty \rightarrow i \rightarrow \frac{\epsilon}{R}, \quad \frac{di}{dt} \rightarrow 0 \)

Time constant for an R-L circuit:

\[ \tau = \frac{L}{R} \]

At \( t = \tau \), the current has risen to \((1-1/e)\) (63 %) of its final value.

Power supplied by the source:

\[ \epsilon \cdot i = i^2 R + L \cdot i \frac{di}{dt} \]

- Power dissipated by R
- Power stored in inductor
Current-Decay in an R-L Circuit

\[ i = I_0 e^{-\left(\frac{R}{L}\right)t} \]

\( t = 0 \rightarrow I_0 \)

\( T = \frac{L}{R} \rightarrow \) for current to decrease to \( 1/e \) (37 % of \( I_0 \)).

Total energy in circuit:

\[ 0 = i^2 R + Li \frac{di}{dt} \]

No energy supplied by a source (no battery present)
5. The L-C Circuit

- In L-C circuit, the charge on the capacitor and current through inductor vary sinusoidally with time. Energy is transferred between magnetic energy in inductor ($U_B$) and electric energy in capacitor ($U_E$). As in simple harmonic motion, total energy remains constant.
L-C Circuit

- $t = 0 \rightarrow$ C charged $\rightarrow Q = C V_m$

- C discharges through inductor. Because of induced emf in L, the current does not change instantaneously. I starts at 0 until it reaches $I_m$.

- During C discharge, the potential in C = induced emf in L. When potential in C = 0 $\rightarrow$ induced emf = 0 $\rightarrow$ maximum $I_m$.

- During the discharge of C, the growing current in L leads to magnetic field $\rightarrow$ energy stored in C (in its electric field) becomes stored in L (in magnetic field).

- After C fully discharged, some i persists (cannot change instantaneously), C charges with contrary polarity to initial state.

- As current decreases $\rightarrow$ B decreases $\rightarrow$ induced emf in same direction as current that slows decrease in current. At some point, B = 0, i = 0 and C fully charged with $-V_m$ (-Q on left plate, contrary to initial state).

- If no energy loses, the charges in C oscillate back and forth infinitely $\rightarrow$ electrical oscillation. Energy is transferred from capacitor E to inductor B.
Electrical Oscillations in an L-C Circuit

Shown is $+ i = dq/dt$ (rate of change of q in left plate). If C discharges $\rightarrow dq/dt < 0 \rightarrow$ counter clockwise “i” is negative.

Kirchhoff’s loop rule: 

$$-L \frac{di}{dt} - \frac{q}{C} = 0$$

$$\frac{d^2 q}{dt^2} + \frac{1}{LC} q = 0$$

L-C circuit

Analogy to eq. for harmonic oscillator: 

$$\frac{d^2 x}{dt^2} + \frac{k}{m} x = 0$$

$$x = A \cos(\omega t + \varphi)$$

$$q = Q \cos(\omega t + \varphi)$$

$$i = \frac{dq}{dt} = -\omega \cdot Q \sin(\omega t + \varphi)$$

Angular frequency of oscillation

$$\omega = \sqrt{\frac{1}{LC}}$$

$$\omega = 2\pi f$$

If at $t = 0 \rightarrow Q_{\text{max}}$ in C, $i = 0 \rightarrow \varphi = 0$

If at $t = 0$, $q=0 \rightarrow \varphi = \pm \pi/2 \text{ rad}$
Energy in an L-C Circuit

Analogy with harmonic oscillator (mass attached to spring):
\[ E_{\text{total}} = 0.5 k A^2 = KE + U_{\text{elas}} = 0.5 m v_x^2 + 0.5 k x^2 \quad (A = \text{oscillation amplitude}) \]

\[ v_x = \pm \sqrt{\frac{k}{m}} \sqrt{A^2 - x^2} \quad \text{Mechanical oscillations} \]

- For L-C Circuit:
\[ \frac{Q^2}{2C} = \frac{1}{2} L i^2 + \frac{q^2}{2C} \]

Total energy initially stored in C = energy stored in L + energy stored in C (at given t).

\[ i = \pm \sqrt{\frac{1}{LC}} \sqrt{Q^2 - q^2} \quad \text{Electrical oscillations} \]
**Mass-Spring System**

Kinetic energy = \( \frac{1}{2} mv_x^2 \)

Potential energy = \( \frac{1}{2} kx^2 \)

\( \frac{1}{2} mv_x^2 + \frac{1}{2} kx^2 = \frac{1}{2} kA^2 \)

\( v_x = \pm \sqrt{k/m} \sqrt{A^2 - x^2} \)

\( v_x = dx/dt \)

\( \omega = \sqrt{\frac{k}{m}} \)

\( x = A \cos(\omega t + \phi) \)

**Inductor-Capacitor Circuit**

Magnetic energy = \( \frac{1}{2} Li^2 \)

Electric energy = \( \frac{q^2}{2C} \)

\( \frac{1}{2} Li^2 + \frac{q^2}{2C} = \frac{Q^2}{2C} \)

\( i = \pm \sqrt{1/LC} \sqrt{Q^2 - q^2} \)

\( i = dq/dt \)
6. The L-R-C Series Circuit

Because of R, the electromagnetic energy of system is dissipated and converted to other forms of energy (e.g. internal energy of circuit materials). [Analogous to friction in mechanical system].

- Energy loses in R $\rightarrow i^2 R \rightarrow U_B$ in L when C completely discharged $< U_E = Q^2/2C$

- Small R $\rightarrow$ circuit still oscillates but with “damped harmonic motion” $\rightarrow$ circuit underdamped.

- Large R $\rightarrow$ no oscillations (die out) $\rightarrow$ critically damped.

- Very large R $\rightarrow$ circuit overdamped $\rightarrow$ C charge approaches 0 slowly.
Analyzing an L-R-C Circuit

\[-iR - L \frac{di}{dt} - \frac{q}{C} = 0\]  
\[\text{(i = dq/dt)}\]

\[\frac{d^2 q}{dt^2} + \frac{R}{L} \frac{dq}{dt} + \frac{1}{LC} q = 0\]

For R < 4L/C: (underdamped)

\[q = Ae^{-(R/2L)t} \cos \left( \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} t + \phi \right)\]

A, \(\phi\) are constants

\[\omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\]

underdamped L-R-C series circuit
Inductors in series: \[ L_{eq} = L_1 + L_2 \]

Inductors in parallel: \[ \frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} \]