PHY 6246 Midterm

11/5/07
p.1

① $V(r) = -\frac{\alpha}{r} - \frac{\beta}{r^2}$ $\quad \alpha>0,\ \beta>0$

(a) $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$

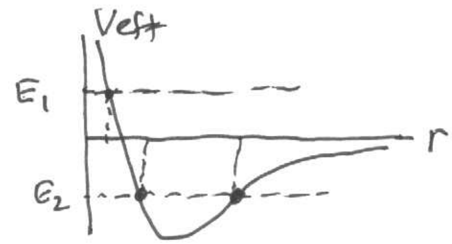
θ cyclic $\Rightarrow$ $\frac{\partial L}{\partial \dot{\theta}} = $ const: $\quad l = mr^2\dot{\theta}$

And $E = T + V = $ const $= \frac{1}{2}m\dot{r}^2 + \underbrace{\frac{l^2}{2mr^2} + V(r)}_{V_{eff}(r) = \frac{l^2}{2mr^2} - \frac{\alpha}{r} - \frac{\beta}{r^2}}$

$$\underline{V_{eff}(r) = -\frac{\alpha}{r} + \frac{b}{r^2} \qquad b = \frac{l^2}{2m} - \beta}$$

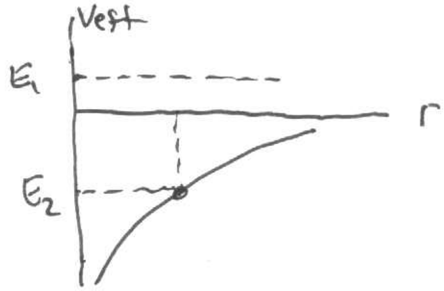
Matters whether $b>0$ or $b<0$

- $\frac{l^2}{2m} > \beta$, $b>0$



- Unbound for $E>0$ (r_{min} where $E = V_{eff}$)
- Bound for $E<0$ (two turning points)

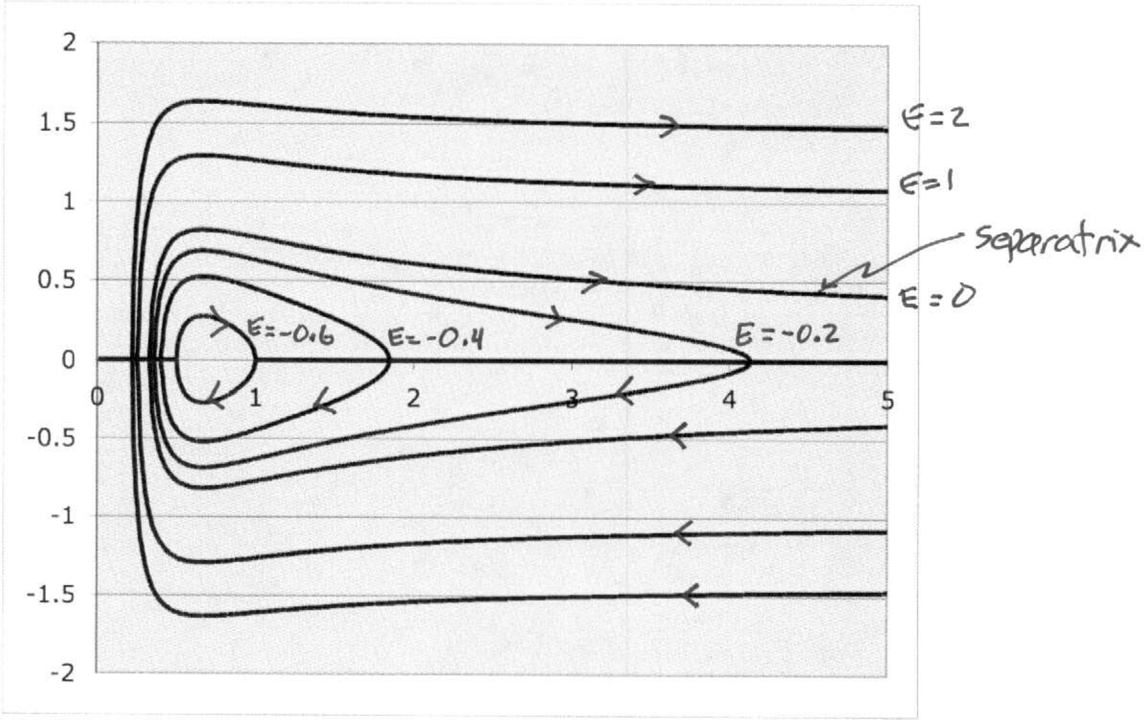
- $\frac{l^2}{2m} < \beta$, $b<0$



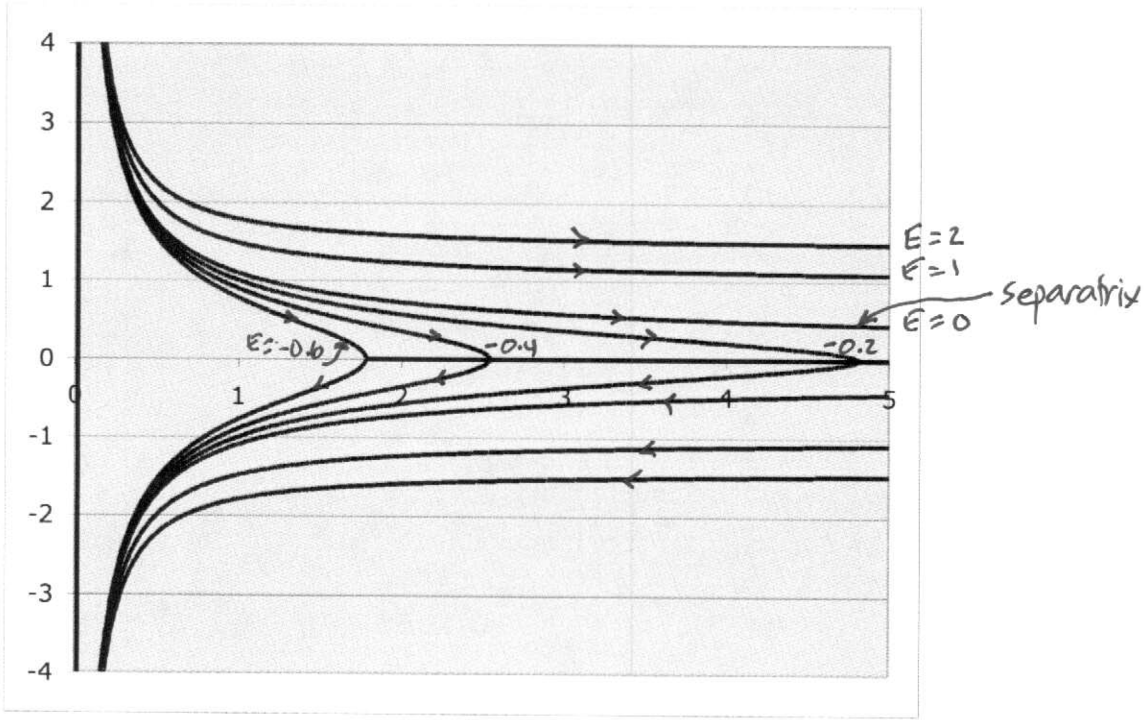
- "Unbound" for $E>0$ (but seems to pass through $r=0$) (see p 3)
- Bound for $E<0$ (but r_{min} appears to be zero?)

(b) Plots are on the next page

Here are some orbits with $l^2/2m>\beta$:



Here are some orbits with $l^2/2m<\beta$:



(c) Constants of motion $\boxed{\begin{aligned} l &= mr^2\dot\theta \\ E &= \tfrac{1}{2}m\dot r^2 + V_{eff}(r)\end{aligned}}$

$$m\ddot r = -\frac{\partial V_{eff}}{\partial r} = \frac{2b}{r^3} - \frac{\alpha}{r^2} \qquad \text{or} \qquad m\ddot r = +\frac{\partial L}{\partial r} \checkmark$$

Have $\dot\theta = \frac{l}{mr^2}$ so if $r \to 0$ get $\dot\theta \to \infty$ (ie for $b<0$)

Also for $b<0$ ~~can~~ have $E = \frac{1}{2}m\dot r^2 + V_{eff}(r)$

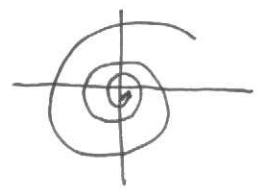
$V_{eff}(r)$ goes to $-\infty$ as $r\to 0$

so $\frac{1}{2}m\dot r^2$ must go to $+\infty$

ie $|\dot r| \to \infty$ as $r \to 0$.

Orbits are peculiar for $b<0$

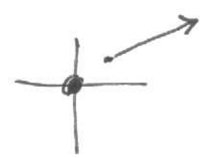
eg $E<0$ (bound) have



Death spiral
$\dot r \to -\infty$
$\dot\theta \to \infty$

$E>0$ ("unbound") depends on initial $\dot r$.

* If $\dot r > 0$, runs away:



But if $\dot r < 0$, spirals in.

(2) Easy to make atlas with two charts:

Chart 1:  x = distance from center to P (omit center)

Chart 2:  y = distance from corner to P (omit corner)

All points on  in a chart $\Rightarrow$  is a manifold.

Easy to see $y = x - R$ for all points in both charts.

This is differentiable $\rightarrow$  is a differentiable manifold.

MpJ

(3) $H = \frac{1}{2}\left(q^{-2} + p^2 q^4\right)$ $\qquad F_2(q\,P) = P/q$

(a) $p = \frac{\partial F_2}{\partial q} = -P/q^2$

$Q = \frac{\partial F_2}{\partial P} = 1/q$ $\qquad$ see 2 p...

so: $\underline{Q = \frac{1}{q} \quad P = -pq^2}$ $\quad$ + $q = \frac{1}{Q},\ p = -P/Q^2$

$K = H + \frac{\partial F_2}{\partial t} = H = \underline{\frac{1}{2}\left[Q^2 + P^2\right] = K}$

(b) $\dot{Q} = \frac{\partial K}{\partial P} = P$

$\dot{P} = -\frac{\partial K}{\partial Q} = -Q$

$$\boxed{\begin{aligned} Q &= Q_0 \cos(t + \varphi) \\ P &= -Q_0 \sin(t + \varphi) \end{aligned}}$$

so $\quad q = \frac{1}{Q_0 \cos(t+\varphi)}$

$p = + \frac{\sin(t+\varphi)}{Q_0 \cos^2(t+\varphi)}$

④ $$H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + \alpha(\underbrace{xp_x + yp_y + zp_z}_{\vec{r}\cdot\vec{p}})^2$$

$$L_z = xp_y - yp_x$$

$$\{H, L_z\} = \left(\frac{\partial H}{\partial x}\frac{\partial L_z}{\partial p_x} - \frac{\partial H}{\partial p_x}\frac{\partial L_z}{\partial x}\right) + \left(\frac{\partial H}{\partial y}\frac{\partial L_z}{\partial p_y} - \frac{\partial H}{\partial p_y}\frac{\partial L_z}{\partial y}\right) + \left(\frac{\partial H}{\partial z}\cancelto{0}{\frac{\partial L_z}{\partial p_z}} - \frac{\partial H}{\partial p_z}\cancelto{0}{\frac{\partial L_z}{\partial z}}\right)$$

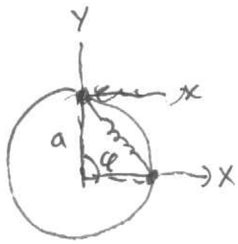
$$= 2\alpha(\vec{r}\cdot\vec{p})p_x(-y) - \left[\frac{p_x}{m} + 2\alpha(\vec{r}\cdot\vec{p})x\right](p_y)$$
$$+ 2\alpha(\vec{r}\cdot\vec{p})p_y(x) - \left[\frac{p_y}{m} + 2\alpha(\vec{r}\cdot\vec{p})y\right](-p_x)$$

$$= 2\alpha(\vec{r}\cdot\vec{p})(\cancel{-p_xy} \cancel{- xp_y} + \cancel{p_yx} + \cancel{yp_x}) + \frac{1}{m}(-p_xp_y + p_yp_x) = 0$$

$$\{H, L_z\} = 0$$

$$\Rightarrow \quad L_z = \text{const.} \qquad \frac{dL_z}{dt} = \cancelto{0}{\{L_z, H\}} + \cancelto{0}{\frac{\partial L_z}{\partial t}}$$

(5)



$x = a\sin\varphi \qquad y = a\cos\varphi$

spring length $\quad \ell^2 = a^2 + a^2 - 2a^2\cos\varphi = 2a^2(1-\cos\varphi)$

$\ell = a\sqrt{2(1-\cos\varphi)}$

spring stretch $\quad x = \ell - a = a\sqrt{2(1-\cos\varphi)} - a$

$\dot x = (a\cos\varphi)\dot\varphi \qquad \dot y = (-a\sin\varphi)\dot\varphi \qquad \dot x^2 + \dot y^2 = a^2\dot\varphi^2$ ✓

$L = \frac{1}{2}ma^2\dot\varphi^2 - mga\cos\varphi - \frac{1}{2}ka^2\left[\cancel{a}\sqrt{2(1-\cos\varphi)} - 1\right]^2$

$\frac{\partial L}{\partial\dot\varphi} = ma^2\dot\varphi$

$\frac{\partial L}{\partial\varphi} = mga\sin\varphi - \cancel{\tfrac{1}{2}}ka^2\cancel{2}\left[\sqrt{2(1-\cos\varphi)} - 1\right]\cancel{\tfrac{1}{2}}\frac{1}{\sqrt{2(1-\cos\varphi)}}(\cancel{2}\sin\varphi)$

$= mga\sin\varphi - ka^2\sin\varphi\left(1 - \frac{1}{\sqrt{2(1-\cos\varphi)}}\right)$

$$\boxed{ma^2\ddot\varphi = mga\sin\varphi - ka^2\sin\varphi\left(1 - \frac{1}{\sqrt{2(1-\cos\varphi)}}\right)} \qquad (*)$$

(b) Equilibrium when $\ddot\varphi = 0 \Rightarrow$ RHS $= 0$

RHS $= a\sin\varphi\left[mg - ka + \frac{ka}{\sqrt{2(1-\cos\varphi)}}\right]$

$= 0$ when $\sin\varphi = 0$ or $[\;] = 0$

$\hookrightarrow \varphi = 0, \pi$

For $[\;] = 0$ use $1-\cos\varphi = 2\sin^2\frac{\varphi}{2} \qquad \sqrt{2(1-\cos\varphi)} = 2\sin\frac{\varphi}{2}$

$[\;] = 0 = mg - ka\left(1 - \frac{1}{2\sin(\varphi/2)}\right) \Rightarrow \frac{mg}{ka} - 1 + \frac{1}{2\sin\varphi/2} = 0$

$$\Rightarrow \boxed{\begin{array}{l}\varphi = 2\sin^{-1}\left[\dfrac{1}{2(1-mg/ka)}\right] \\ \text{or } \varphi = 0 \text{ or } \varphi = \pi\end{array}}$$

Expect $\varphi = 0$ is never stable: gravity & spring both move away.

Expect $\varphi = \pi$ is stable for <u>weak</u> spring.

~~Expect~~
Strong spring would have $\varphi \approx 60°$ ~~tko~~
ie $k \to \infty$ (like a rigid bar)

check my angle: when $k \to \infty$ $\varphi = 2\sin^{-1}\frac{1}{2} = 60°$ ✓

so expect $\varphi = 2\sin^{-1}\left[\frac{1}{2(1 - mg/ka)}\right]$ for sufficiently strong spring.

For this to be possible at all need $\frac{1}{2(1-mg/ka)} < 1 \Rightarrow$ <u>$ka > 2mg$</u>

I expect this to determine whether the equilibrium is $\varphi = \pi$ or $\varphi = 2\sin^{-1}(\;''\;)$.

(c) (i) say $\varphi = \pi \pm x$

$\cos\varphi = -\cos x \approx -1 + \frac{x^2}{2}$ $\quad \sin\varphi \approx \sin x \approx x$

Look back at $(*)$

$$-ma^2\ddot{x} = (mga)x - ka^2x\left(1 - \frac{1}{\sqrt{2(2 - x^2/2)}}\right) + O(x^3)$$

drop: $O(x^3)$

$$\approx mgax - ka^2x\left(1 - \tfrac{1}{2}\right)$$
$$= \left(mga - \tfrac{1}{2}ka^2\right)x$$

~~$0 = \ddot{x} + \frac{1}{a}(\ldots - \frac{ka}{2m})$~~ $\quad 0 = \ddot{x} + \frac{1}{ma}\left(mg - \frac{1}{2}ka\right)x$

~~$0 = \frac{1}{a}k$~~

Get stable oscillation if $mg > \frac{1}{2}ka$

ie <u>$ka < 2mg$</u> ✓

and see $\omega = \sqrt{\frac{1}{ma}\left(mg - \frac{1}{2}ka\right)} = \sqrt{\frac{g}{a} - \frac{k}{2m}}$

(ii) say $\varphi = \varphi_0 + x$ where $\varphi_0 = 2\sin^{-1}\left[\frac{1}{2(1-mg/ka)}\right]$

RHS of $*$ is $\sin\varphi\left[mga - ka^2\left(1 - \frac{1}{\sqrt{2(1-\cos\varphi)}}\right)\right]$

$= a\sin\varphi\left[mg - ka\left(1 - \frac{1}{2\sin\varphi/2}\right)\right]$

$\sin\varphi = \sin(\varphi_0 + x) = \sin\varphi_0 \underbrace{\cos x}_{1+O(x^2)} + \cos\varphi_0 \underbrace{\sin x}_{x + O(x^3)}$

$\simeq \sin\varphi_0 + x\cos\varphi_0$

$\sin\frac{\varphi}{2} = \sin\left(\frac{\varphi_0}{2} + \frac{x}{2}\right) \approx \sin\frac{\varphi_0}{2} + \frac{x}{2}\cos\frac{\varphi_0}{2}$

$\frac{1}{2\sin\frac{\varphi}{2}} = \frac{1}{2\sin\frac{\varphi_0}{2} + x\cos\frac{\varphi_0}{2}} = \frac{1}{2\sin\frac{\varphi_0}{2}}\frac{1}{1+\frac{1}{2}x\cot\frac{\varphi_0}{2}} =$

$= \frac{1}{2\sin\frac{\varphi_0}{2}}\left(1 - \frac{1}{2}x\cot\frac{\varphi_0}{2}\right)$ + higher ord

$RHS \approx a\left[\sin\varphi_0 + x\cos\varphi_0\right]\left[mg - ka\left(1 - \frac{1}{2\sin\frac{\varphi_0}{2}}\right) - ka\frac{x\cot\varphi_0/2}{4\sin\varphi_0/2}\right]$

$mg - ka\left(1 - \frac{1}{2\sin\varphi_0/2}\right) = 0$

$RHS \simeq -ka^2(\sin\varphi_0 + x\cos\varphi_0)\frac{x\cot\varphi_0/2}{4\sin\varphi_0/2}$

drop!

$\simeq -ka^2\, 2\cancel{\sin\frac{\varphi_0}{2}}\cos\frac{\varphi_0}{2}\frac{x\cos\varphi_0/2}{4\sin^2\varphi_0/2} = \left(ka^2\frac{\cos^2\varphi_0/2}{\sin\varphi_0/2}\right)x$

$* \Rightarrow \quad ma^2\ddot{x} = -ka^2\left(\frac{\cos^2\varphi_0/2}{\sin\varphi_0/2}\right)x$

$\Rightarrow \quad \ddot{x} + \frac{k}{m}\frac{\cos^2\varphi_0/2}{\sin\varphi_0/2}x = 0$

Get stable oscillation with $\omega = \sqrt{\frac{k}{m}\frac{\cos^2\varphi_0/2}{\sin\varphi_0/2}}$ if $\sin\frac{\varphi_0}{2} > 0$
and of course φ_0 = real
$\Rightarrow ka > 2mg$

$\omega = \sqrt{\frac{k}{m}2\left(1-\frac{mg}{ka}\right)\left[1 - \frac{1}{4(1-mg/ka)^2}\right]}$ if $[\,] > 0 \Rightarrow ka \geq 2mg$

(6) constraint : $\rho = a$ or $f(\rho) = \rho - a = 0$

$\vec{\nabla} f = \hat{\rho}$

$$L = \tfrac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2) - mg\rho\cos\varphi - \tfrac{1}{2}k\left[(a^2+\rho^2-2a\rho\cos\varphi)^{1/2} - a\right]^2 + \lambda(\rho - a)$$

$$\frac{\partial L}{\partial \dot{\rho}} = m\dot{\rho}$$

$$\frac{\partial L}{\partial \rho} = m\rho\dot{\varphi}^2 - mg\cos\varphi - k\left(\sqrt{a^2+\rho^2-2a\rho\cos\varphi} - a\right)\frac{\rho - a\cos\varphi}{\sqrt{a^2+\rho^2-2a\rho\cos\varphi}} + \lambda$$

$$\Rightarrow \quad m\ddot{\rho} = \quad " \quad " \quad "$$

Constraint $\Rightarrow$ $\ddot{\rho} = 0$ $\quad \rho = a$

$$0 = ma\dot{\varphi}^2 - mg\cos\varphi - k\left(a\sqrt{2(1-\cos\varphi)} - a\right)\frac{a(1-\cos\varphi)}{a\sqrt{2(1-\cos\varphi)}} + \lambda$$

$$0 = ma\dot{\varphi}^2 - mg\cos\varphi - \tfrac{1}{2}ka\left(\sqrt{2(1-\cos\varphi)} - 1\right)\sqrt{2(1-\cos\varphi)} + \lambda$$

$$\lambda = -ma\dot{\varphi}^2 + mg\cos\varphi + k\left(a\sqrt{2(1-\cos\varphi)} - a\right)\sin\frac{\varphi}{2}$$

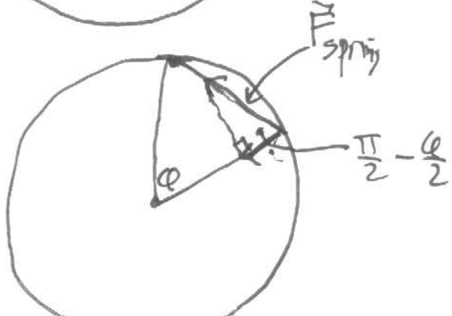
$$\boxed{\vec{C} = \lambda\hat{\rho} = \left[-ma\dot{\varphi}^2 + mg\cos\varphi + k\left(a\sqrt{2(1-\cos\varphi)} - a\right)\sin\frac{\varphi}{2}\right]\hat{\rho}}$$

should be easy to understand.

The radial component of gravity is $mg\cos\varphi$ inward

$\Rightarrow$ need $(mg\cos\varphi)\hat{\rho}$ to cancel ✓

The last term is $\underbrace{k\cdot(\text{stretch of spring})}_{\text{force of spring}} \cdot \sin\frac{\varphi}{2}$

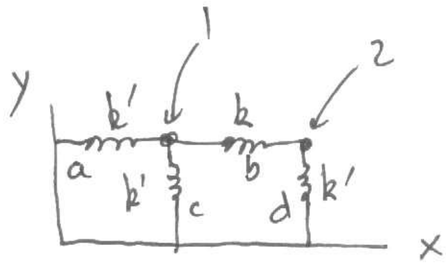


The radial component of this is $(F_{spring})\cos(\frac{\pi}{2} - \frac{\varphi}{2})$

$= F_{spring}\sin\frac{\varphi}{2}$ inward

So the last term in $\vec{C}$ cancels the radial component of the spring force.

and The 1st term in $\vec{C}$ produces the needed centripetal acceleration ✓



(7)

Measure $x_1 y_1$ $x_2 y_2$ from equilibrium
"actual" pos^n x_1+l y_1+l
x_2+2l y_2+l

length of spring $a = \sqrt{(l+x_1)^2+y_1^2}$

length of spring $b = \sqrt{(l+x_2-x_1)^2+(y_2-y_1)^2}$

" " $c = \sqrt{x_1^2+(l+y_1)^2}$

" " $d = \sqrt{x_2^2+(l+y_2)^2}$

$$V = \tfrac12 k'\left[\sqrt{(l+x_1)^2+y_1^2}-l\right]^2 + \tfrac12 k'\left[\sqrt{x_1^2+(l+y_1)^2}-l\right]^2 + \tfrac12 k'\left[\sqrt{x_2^2+(l+y_2)^2}-l\right]^2$$
$$+ \tfrac12 k\left[\sqrt{(l+x_2-x_1)^2+(y_2-y_1)^2}-l\right]^2$$

$$T = \tfrac12 m(\dot x_1^2+\dot y_1^2+\dot x_2^2+\dot y_2^2)$$

$$L = T - V$$

(b) Cookbook. Expand V for small x+y

$$\sqrt{(l+a)^2+b^2} \approx (l+a)\left[1+\frac{b^2}{(l+a)^2}\right]^{1/2} \approx (l+a)\left[1+\frac12\frac{b^2}{l^2}\right] + O(3)$$
$$= l + a + b^2/2l + O(3) \qquad a, b = \text{small}$$

$$(\sqrt{\;''\;}-l)^2 \approx (a+b^2/2l)^2 \approx a^2$$

$\cancel{V \approx \tfrac12 k'}$

$\tfrac12 k(x_1^2+x_2^2-2x_1x_2)$

$$V \approx \tfrac12 k' x_1^2 + \tfrac12 k' y_1^2 + \tfrac12 k' y_2^2 + \tfrac12 k(x_2-x_1)^2$$

$$= \tfrac12 (x_1\ x_2\ y_1\ y_2)\underbrace{\begin{pmatrix} k'+k & -k & 0 & 0 \\ -k & k'+k & 0 & 0 \\ 0 & 0 & k' & 0 \\ 0 & 0 & 0 & k' \end{pmatrix}}_{K}\begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix}$$

and $$T = \tfrac12(\dot x_1\ \dot x_2\ \dot y_1\ \dot y_2)\ \underbrace{m\mathbb{1}}_{M}\begin{pmatrix} \dot x_1 \\ \vdots \\ \dot y_2 \end{pmatrix}$$

solve $(K-\omega^2 M)\underline{a} = 0$ put $\lambda = m\omega^2$

$$\begin{pmatrix} k+k'-\lambda & -k & 0 & 0 \\ -k & k-\lambda & 0 & 0 \\ 0 & 0 & k'-\lambda & 0 \\ 0 & 0 & 0 & k'-\lambda \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = 0$$

$$\det(\;) = 0 = (k'-\lambda)^2 \begin{vmatrix} k+k'-\lambda & -k \\ -k & k'-\lambda \end{vmatrix} = (k'-\lambda)^2\left[(k+k'-\lambda)(k'-\lambda) - k^2\right]$$

$$= (\lambda-k')^2\left[(\lambda-k)(\lambda-k'-k) - k^2\right] = 0$$

$$= (\lambda-k')^2\left[\lambda^2 - \lambda(2k+k') + (k^2 + k'k - k^2)\right] = 0$$

$$= (\lambda-k')^2\left[\lambda^2 - (2k+k') + kk'\right] = 0$$

so $\lambda = k'$ (twice) or $\lambda = \frac{1}{2}\left[2k+k' \pm \sqrt{4k^2 + \cancel{4kk'} + k'^2 - \cancel{4kk'}}\right]$

$\lambda = k + \frac{1}{2}k' \pm \sqrt{k^2 + k'^2/4}$

1) $\lambda_1 = k + \frac{1}{2}k' + \sqrt{k^2 + k'^2/4}$

$$\begin{pmatrix} \frac{1}{2}k' - \sqrt{\;} & -k & 0 & 0 \\ -k & -\frac{1}{2}k' - \sqrt{\;} & 0 & 0 \\ 0 & 0 & \frac{1}{2}k' - k - \sqrt{\;} & 0 \\ 0 & 0 & 0 & \frac{1}{2}k' - k - \sqrt{\;} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = 0$$

$a_3 = a_4 = 0$ and $\left(\frac{1}{2}k' - \sqrt{\;}\right) a_1 = k a_2 \Rightarrow a_1 = \dfrac{k}{\frac{1}{2}k' - \sqrt{k^2 + k'^2/4}}\, a_2$

$$\omega_1 = \sqrt{\frac{\lambda_1}{m}} = \sqrt{\frac{1}{m}\left(k + \frac{1}{2}k' + \sqrt{k^2 + k'^2/4}\right)}$$

$$\underline{x}^{(1)} = e^{i\omega_1 t}\begin{pmatrix} \dfrac{k}{\frac{1}{2}k' - \sqrt{k^2 + k'^2/4}} \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

2) $\lambda_2 = k + \frac{1}{2}k' - \sqrt{k^2 + k'^2/4}$ same with $\sqrt{\ } \to -\sqrt{\ }$

$\omega_2 = \sqrt{\frac{\lambda_2}{m}} = \sqrt{\frac{1}{m}\left(k + \frac{1}{2}k' - \sqrt{k^2 + k'^2/4}\right)}$

$$\underset{\sim}{x}^{(2)} = e^{i\omega_2 t} \begin{pmatrix} \dfrac{k}{\frac{1}{2}k' + \sqrt{k^2 + k'^2/4}} \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

3) & 4) $\lambda_3 = \lambda_4 = k'$

$$\begin{pmatrix} k & -k & 0 & 0 \\ -k & k-k' & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

a_3 & a_4 = anything.

$a_1 = a_2$

$-ka_1 + (k-k')a_1 = 0 \Rightarrow a_1 = 0$

so $a_2 = 0$

$\omega_3 = \omega_4 = \sqrt{\frac{k'}{m}}$

$$x^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{i\omega_3 t} \qquad x^{(4)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{i\omega_4 t}$$

(c) $x_1 = y_1 = 0 \qquad x_2 = y_2 = \epsilon/\sqrt{2}$

$\dot{x}_1 = \dot{y}_1 = \dot{x}_2 = \dot{y}_2 = 0$

Write $\underset{\sim}{x} = \sum_{j=1}^{4} c_j e^{i\omega_j t} \underset{\sim}{a}^{(j)} + cc = \cancel{\sum_4 c_j + c}$

$\cancel{\dot{\underset{\sim}{x}} = \sum_{j=1}^{4} i\omega_j (c_j - c_j^*) e^{i\omega_j}}$

$$\dot{\underset{\sim}{x}} = \sum_{j=1}^{4} \left(c_j e^{i\omega_j t} - c_j^* e^{-i\omega_j t}\right) i\omega_j \underset{\sim}{a}^{(j)}$$

$$\underset{\sim}{x}(0) = \sum_j (c_j + c_j^*) \underset{\sim}{a}^{(j)} = \begin{pmatrix} 0 \\ \epsilon/\sqrt{2} \\ 0 \\ \epsilon/\sqrt{2} \end{pmatrix}$$

$$\dot{\underset{\sim}{x}}(0) = \sum_j (c_j - c_j^*) i\omega_j \underset{\sim}{a}^{(j)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$2^d \Rightarrow \quad c_j = c_j^* \quad$ ie $\quad c_j =$ real

$$1^{st} \Rightarrow \quad c_1 \begin{pmatrix} \frac{k}{\frac{1}{2}k' - \sqrt{\ }} \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} \frac{k}{\frac{1}{2}k' + \sqrt{\ }} \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \frac{\epsilon}{2\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\left.\begin{aligned} c_1 \frac{k}{\frac{1}{2}k' - \sqrt{\ }} + c_2 \frac{k}{\frac{1}{2}k' + \sqrt{\ }} &= 0 \\ c_1 + c_2 &= \epsilon/2\sqrt{2} \end{aligned}\right\} \quad \begin{aligned} c_1 &= \frac{\epsilon}{2\sqrt{2}} \frac{k^2}{2\sqrt{k^2 + k'^2/4}\left(\frac{1}{2}k' + \sqrt{\ }\right)} \\ c_2 &= -\frac{\epsilon}{2\sqrt{2}} \frac{k^2}{2\sqrt{\ }\left(\frac{1}{2}k' - \sqrt{\ }\right)} \end{aligned}$$

$$c_3 = 0$$

$$c_4 = \epsilon/2\sqrt{2}$$

$$\boxed{\text{so} \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} = \frac{\epsilon}{\sqrt{2}} \left\{ \frac{k^2}{2\sqrt{\ }\left(\frac{1}{2}k' + \sqrt{\ }\right)} \begin{pmatrix} \frac{k}{\frac{1}{2}k' - \sqrt{\ }} \\ 1 \\ 0 \\ 0 \end{pmatrix} \cos\omega_1 t - \frac{k^2}{2\sqrt{\ }\left(\frac{1}{2}k' - \sqrt{\ }\right)} \begin{pmatrix} \frac{k}{\frac{1}{2}k' + \sqrt{\ }} \\ 1 \\ 0 \\ 0 \end{pmatrix} \cos\omega_2 t + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \cos\omega_4 t \right\}}$$

MpD

(d) Take $k' \to 0$

mode 1 $\quad \lambda_1 = 2k \quad \omega_1 = \sqrt{\frac{2k}{m}} = \sqrt{\frac{k}{\mu}} \quad \underset{\sim}{a}^{(1)} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ stretch this spring

reduced mass $\mu = m/2$

mode 2 $\quad \lambda_2 = 0 \quad \omega_2 = 0 \quad \underset{\sim}{a}^{(2)} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ move together

mode 3 & mode 4 ~~unchanged~~ $\omega_3 = \omega_4 = 0 \quad a^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad a^{(4)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

For (c): $\quad \sqrt{\ } = k$

$$\begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} = \frac{\epsilon}{\sqrt{2}} \left\{ \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \cos\omega_1 t + \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\boxed{\begin{aligned} x_1 &= \frac{\epsilon}{\sqrt{2}} \frac{1}{2}(1 - \cos\omega_1 t) \\ x_2 &= \frac{\epsilon}{\sqrt{2}} \frac{1}{2}(1 + \cos\omega_1 t) \\ y_1 &= 0 \\ y_3 &= \epsilon/\sqrt{2} \end{aligned}}$$

(8) $L = \frac{1}{2}(\dot{q}^2 - \omega^2 q^2) - \alpha q^3$

(a) $p = \frac{\partial L}{\partial \dot{q}} = \dot{q}$

$H = p\dot{q} - L = \frac{p^2}{2} + \frac{1}{2}\omega^2 q^2 + \alpha q^3$

(b) $F_3(pQ) = pQ + apQ^2 + bp^2Q + cp^3 + dQ^3$

$q = -\frac{\partial F_3}{\partial p} = -(Q + aQ^2 + 2bpQ + 3cp^2)$ (1)

$P = -\frac{\partial F_3}{\partial Q} = -(p + 2apQ + bp^2 + 3dQ^2)$ (2)

Need q, p in terms of Q, P

Key fact: only need these to order 2 - because need H to order 3.

eg. If $q = -Q + \gamma Q^2$ (which it does not!) $+ O(3)$
then $q^2 = Q^2 - 2\gamma Q^3 + O(4)$

(2) $\Rightarrow$ $-P = (1+2aQ)p + bp^2 + 3dQ^2$
$0 = bp^2 + (1+2aQ)p + (P + 3dQ^2)$

$\Rightarrow p = \frac{1}{2b}\left[-(1+2aQ) \pm \sqrt{(1+2aQ)^2 - 4b(P+3dQ^2)}\right]$

$\sqrt{\ } = (1 + 4aQ + 4a^2Q^2 - 4bP - 12bdQ^2)^{1/2}$
$= [1 + 4aQ + (4a^2 - 12bd)Q^2 - 4bP]^{1/2}$
$= [1 + (4aQ - 4bP) + (4a^2 - 12bd)Q^2]^{1/2}$

keep to $O(2)$ (ie drop $O(3)$): $(1+x)^{\frac{1}{2}} \simeq 1 + \frac{1}{2}x - \frac{1}{8}x^2$

$\sqrt{\ } \simeq 1 + \frac{1}{2}(4aQ - 4bP) + \frac{1}{2}(4a^2 - 12bd)Q^2 - \frac{1}{8}16(aQ - bP)^2 + O(3)$

$\sqrt{\ } \simeq 1 + 2(aQ - bP) + (2a^2 - 6bd)Q^2 - 2(aQ - bP)^2$
$= 1 + (2aQ - 2bP) + (\cancel{2a^2} - 6bd)Q^2 - \cancel{2a^2Q^2} + 4abQP - 2b^2P^2$

$\sqrt{\ } = 1 + (2aQ - 2bP) + (-6bdQ^2 + 4abQP - 2b^2P^2) + O(3)$

see need ~~neg.~~ pos sign in $p = \frac{1}{2b}[-(1+2aQ) + \sqrt{\ }]$

$p = \frac{1}{2b}[-\cancel{1} - 2aQ + \cancel{1} + (2aQ - 2bP) + (-6bdQ^2 + 4abQP - 2b^2P^2)] + O(3)$

$$p = -P + (-3dQ^2 + 2aQP - bP^2) + O(3)$$

$$p = -[P + (3dQ^2 - 2aQP + bP^2)] + O(3)$$

Plug this into (1) + keep to order O(2):

$$q = -[Q + (aQ^2 - 2bPQ + 3cP^2)] + O(3)$$

Now $p^2 = P^2 + (6dQ^2P - 4aQP^2 + 2bP^3) + O(4)$

$q^2 = Q^2 + (2aQ^3 - 4bPQ^2 + 6cQP^2) + O(4)$

$q^3 = Q^3 + O(4)$

$$K = H + \frac{\partial F_2}{\partial t} = H$$

$$K = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 + \alpha q^3$$

$$= \frac{1}{2}P^2 + (3dQ^2P - 2aQP^2 + bP^3) + \frac{1}{2}\omega^2 Q^2 + \omega^2(aQ^3 - 2bQ^2P + 3cQP^2) + \alpha Q^3 + O(4)$$

$$K = \frac{1}{2}P^2 + \frac{1}{2}\omega^2 Q^2 + \left[(a\omega^2 + \alpha)Q^3 + (3d - 2b\omega^2)Q^2P + (-2a + 3c\omega^2)QP^2 + bP^3\right] + O(4)$$

Want all terms in [ ] to vanish

$bP^3 = 0 \Rightarrow \boxed{b = 0}$

then $(3d - 2b\omega^2)Q^2P = 0 \Rightarrow \boxed{d = 0}$

$(a\omega^2 + \alpha)Q^3 = 0 \Rightarrow a = -\frac{\alpha}{\omega^2}$

then $(-2a + 3c\omega^2)QP^2 = 0 \Rightarrow c = -\frac{2a}{3\omega^2} = \frac{2\alpha}{3\omega^4}$

then $\boxed{K = \frac{1}{2}P^2 + \frac{1}{2}\omega^2 Q^2}$

mp18

Then $$q = -\left[Q \mp \frac{\alpha}{\omega^2}Q^2 + \frac{2\alpha}{\omega^4}P^2\right] \quad + O(3)$$
$$p = -\left[P + \frac{2\alpha}{\omega^2}QP\right] \quad + O(3)$$

I could go back and keep the RHS through O(3) ... I won't bother.

Solve: $$Q = Q_0 \cos(\omega t + \varphi)$$
$$P = -\omega Q_0 \sin(\omega t + \varphi)$$

$$\Rightarrow \quad q(t) = -Q_0\cos(\omega t + \varphi) \pm \frac{\alpha Q_0^2}{\omega^2}\left[\cos^2(\omega t + \varphi) + 2\sin^2(\omega t + \varphi)\right]$$

$$q(t) = -Q_0\cos(\omega t + \varphi) \pm \frac{\alpha Q_0^2}{\omega^2}\left(1 + \sin^2(\omega t + \varphi)\right)$$ $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$$q(t) = -Q_0\cos(\omega t + \varphi) \pm \frac{\alpha Q_0^2}{2\omega^2}\left(3 - \cos(2\omega t + 2\varphi)\right)$$

get $\omega \pm \omega$ because of αq^3 term in H

$$p(t) = -P + \frac{2\alpha}{\omega^2}QP$$
$$= -\omega Q_0 \sin(\omega t + \varphi) \mp \frac{2\alpha}{\omega^2}Q_0(-\omega Q_0)\sin(\omega t + \varphi)\cos(\omega t + \varphi)$$

$$p(t) = -\omega Q_0 \sin(\omega t + \varphi) \pm \frac{\alpha Q_0^2}{\omega}\sin(2\omega t + 2\varphi)$$