Vector Spaces

A **linear vector space** $\mathbb{V}$ is a set of objects (the vectors) along with definite rules for vector addition and scalar multiplication which satisfy the following rules. (Here the vectors are denoted by lower-case Roman letters $\mathrm{a}, \mathrm{b}, \mathrm{c}, \ldots$ and scalars by lower-case Greek letters $\alpha, \beta, \ldots$.)

1. **Vector addition**, denoted $\mathrm{a} + \mathrm{b}$, must satisfy the following:

   (a) Closure: $\mathrm{a} + \mathrm{b} \in \mathbb{V}$

   (b) Commutivity: $\mathrm{a} + \mathrm{b} = \mathrm{b} + \mathrm{a}$

   (c) Associativity: $(\mathrm{a} + \mathrm{b}) + \mathrm{c} = \mathrm{a} + (\mathrm{b} + \mathrm{c})$

   (d) Identity: there exists a null vector $\varnothing$ for which $\mathrm{a} + \varnothing = \mathrm{a}$ for all a

   (e) For every vector a there exists an additive inverse, denoted $-\mathrm{a}$, for which $\mathrm{a} + (-\mathrm{a}) = \varnothing$

2. **Scalar multiplication**, denoted $\alpha\mathrm{a}$, must satisfy the following:

   (a) Closure: $\alpha\mathrm{a} \in \mathbb{V}$

   (b) Associativity: $\alpha(\beta\mathrm{c}) = (\alpha\beta)\mathrm{c}$

   (c) Distributivity in vectors: $\alpha(\mathrm{a} + \mathrm{b}) = \alpha\mathrm{a} + \alpha\mathrm{b}$

   (d) Distributivity in scalars: $(\alpha + \beta)\mathrm{a} = \alpha\mathrm{a} + \beta\mathrm{a}$

   (e) Identity: $1\mathrm{a} = \mathrm{a}$

Inner Product Spaces

Inner products can be denoted (a, b) or $\mathrm{a} \cdot \mathrm{b}$ or (my preference) $\langle \mathrm{a}|\mathrm{b} \rangle$.

An **inner product** $\langle \mathrm{a}|\mathrm{b} \rangle$ of two vectors is a definite rule that produces a scalar and satisfies:

$$\begin{aligned} \langle \mathrm{a}|\mathrm{b} \rangle &= \langle \mathrm{b}|\mathrm{a} \rangle^* \\ \langle \mathrm{a}|\beta\mathrm{b} + \gamma\mathrm{c} \rangle &= \beta\langle \mathrm{a}|\mathrm{b} \rangle + \gamma\langle \mathrm{a}|\mathrm{c} \rangle \end{aligned}$$

The only useful inner products in Physics are also positive semi-definite, which means:

$$\langle \mathrm{a}|\mathrm{a} \rangle \geq 0, \quad \text{and} \quad \langle \mathrm{a}|\mathrm{a} \rangle = 0 \quad \text{iff} \quad \mathrm{a} = \varnothing.$$