1. The operator $L = \frac{i}{\hbar} \frac{d}{dx}$ has as its domain the set of all differentiable functions on the interval $0 \leq x \leq L$ with periodic boundary conditions $f(0) = f(L)$. Derive the adjoint operator $L^\dagger$ and its domain.

Solution:

$$\langle g | L f \rangle = \int_0^L dx \, g^*(x) \frac{df}{dx}$$

$$= i [g^*(L)f(L) - g^*(0)f(0)] - i \int_0^L dx \, \frac{dg^*}{dx} \frac{df}{dx}$$

$$= i f(0) [g(L) - g(0)] + \int_0^L dx \left( \frac{dg}{dx} \right)^* f.$$ 

If $g(L) = g(0)$ the boundary term vanishes. This defines the domain of the adjoint operator. For such functions $g$ the above becomes

$$\langle g | L f \rangle = \langle L^\dagger g | f \rangle$$

with $L^\dagger = \frac{i}{\hbar} \frac{d}{dx}$. Since $L^\dagger = L$ and the domains are identical, $L$ is self-adjoint.

2. The operator $L = \frac{i}{\hbar} \frac{d}{dx}$ has as its domain the set of all differentiable functions on the interval $0 \leq x \leq L$ with boundary conditions $f'(0) = 0 = f'(L)$ (notice the two derivatives). Derive the adjoint operator $L^\dagger$ and its domain.

Solution:

$$\langle g | L f \rangle = \int_0^L dx \, g^*(x) \frac{df}{dx}$$

$$= i [g^*(L)f(L) - g^*(0)f(0)] - i \int_0^L dx \, \frac{dg^*}{dx} \frac{df}{dx}$$

Since $f(L)$ and $f(0)$ can be anything, the boundary term only vanishes if $g(L) = 0 = g(0)$. This defines the domain of the adjoint operator. For such functions $g$ the above becomes

$$\langle g | L f \rangle = \langle L^\dagger g | f \rangle$$

with $L^\dagger = \frac{i}{\hbar} \frac{d}{dx}$. We have $L^\dagger = L$; but because the domains are different, $L$ is not self-adjoint. (In fact it is not even Hermitian: the boundary term does not vanish for all functions $g$ in the original domain.)
3. The operator $\mathcal{L} = \frac{d^2}{dx^2}$ has as its domain the set of all twice-differentiable functions on the interval $0 \leq x \leq L$ with boundary conditions $f'(0) = 0 = f(L)$ (notice the one derivative). Derive the adjoint operator $\mathcal{L}^\dagger$ and its domain.

**Solution:**

$$\langle g|\mathcal{L}f \rangle = \int_0^L dx \, g^*(x) \frac{d^2f}{dx^2}$$

$$= \left[ g^*(L) \frac{df}{dx} \bigg|_{x=L} - g^*(0) \frac{df}{dx} \bigg|_{x=0} \right] - \int_0^L dx \, \frac{dg^*}{dx} \frac{df}{dx}$$

$$= g^*(L) \frac{df}{dx} \bigg|_{x=L} - \int_0^L dx \, \frac{dg^*}{dx} \frac{df}{dx}.$$

Here part of the boundary term vanishes because of the boundary condition $f'(0) = 0$. Integrating by parts again gives

$$\langle g|\mathcal{L}f \rangle = g^*(L) \frac{df}{dx} \bigg|_{x=L} - \left[ \frac{dg^*}{dx} \bigg|_{x=L} f(L) - \frac{dg^*}{dx} \bigg|_{x=0} f(0) \right] + \int_0^L dx \, \frac{d^2g^*}{dx^2} f$$

$$= g^*(L) \frac{df}{dx} \bigg|_{x=L} + \frac{dg^*}{dx} \bigg|_{x=0} f(0) + \int_0^L dx \, \left( \frac{d^2g}{dx^2} \right)^* f,$$

where I have used $f(L) = 0$. The boundary term vanishes if $g'(0) = 0 = g(L)$. This defines the domain of the adjoint operator. For such functions $g$ the above becomes

$$\langle g|\mathcal{L}f \rangle = \langle \mathcal{L}^\dagger g|f \rangle$$

with $\mathcal{L}^\dagger = \frac{d^2}{dx^2}$. Since $\mathcal{L}^\dagger = \mathcal{L}$ and the domains are identical, $\mathcal{L}$ is self-adjoint.

4. Find the eigenfunctions and eigenvalues of the operator $\mathcal{L} = i \frac{d}{dx}$ on the interval $[0,L]$ with periodic boundary conditions $y(L) = y(0)$. Show that, properly normalized, the eigenfunctions form an infinite orthonormal set.

**Solution:** The eigenfunction ODE

$$i \frac{dy}{dx} = \lambda y$$

has the general solution $y(x) = Ce^{-i \lambda x}$. Imposing $y(L) = y(0)$ yields $e^{-i \lambda L} = 1$. This is solved by $\lambda L = 2n\pi$ with $n = 0, \pm 1, \pm 2, \ldots$. Thus the eigenvalues are

$$\lambda_n = \frac{2n\pi}{L}, \quad n = 0, \pm 1, \pm 2, \ldots.$$
The eigenfunctions are \( y_n(x) = Ce^{-i2n\pi x/L} \). These are normalized by setting

\[
\langle y_n | y_n \rangle = 1 = \int_0^L \! dx \ y_n^*(x) y_n(x) = \int_0^L \! dx \ |C|^2 = |C|^2 L,
\]

which is solved by \( C = 1/\sqrt{L} \). Thus the normalized eigenfunctions are

\[
y_n(x) = \frac{1}{\sqrt{L}} e^{-i2n\pi x/L}.
\]

Orthogonality is checked by examining

\[
\langle y_n | y_m \rangle = \frac{1}{L} \int_0^L \! dx \ e^{i2\pi(n-m)/L}.
\]

When \( m = n \) this produces unity, as above. When \( m \neq n \) it gives

\[
\frac{1}{L} \int_0^L \! dx \ e^{i2\pi(n-m)/L} \bigg|_0^L = \frac{1}{i2\pi(n-m)} \left[ e^{i2\pi(n-m)} - 1 \right] = 0.
\]

The last step follows because \( e^{i2\pi k} = 1 \) for any integer \( k \).

5. Find the eigenfunctions and eigenvalues of the operator \( \mathcal{L} = i \frac{d}{dx} \) on the interval \([0,L]\) with boundary conditions \( y'(0) = 0 = y'(L) \) (notice the two derivatives).

**Solution:** The eigenfunction ODE

\[
i \frac{dy}{dx} = \lambda y
\]

has the general solution \( y(x) = Ce^{-i\lambda x} \). Imposing \( y'(0) = 0 \) yields

\[-iC\lambda = 0.
\]

Choosing \( C = 0 \) gives the trivial solution \( y(x) \equiv 0 \). The other choice \( \lambda = 0 \) gives \( y(x) = C \). In that case the other boundary condition \( y'(L) = 0 \) is also satisfied. Thus we have managed to find only one eigenstate:

\[
\lambda = 0, \quad y(x) = \text{const}.
\]

We did not find an infinite set because the operator is not self-adjoint; briefly, because the boundary conditions are unsuitable.
6. Find the eigenfunctions and eigenvalues of the operator \( \mathcal{L} = \frac{d^2}{dx^2} \) on the interval \([0,L]\) with boundary conditions \( y'(0) = 0 = y(L) \) (notice the one derivative). Show that, properly normalized, the eigenfunctions form an infinite orthonormal set.

Solution: The eigenfunction ODE

\[
\frac{d^2 y}{dx^2} = \lambda y
\]

has the general solution

\[
y(x) = A \sinh(\lambda^{\frac{1}{2}} x) + B \cosh(\lambda^{\frac{1}{2}} x).
\]

Taking one derivative gives

\[
y'(x) = A \lambda^{\frac{1}{2}} \cosh(\lambda^{\frac{1}{2}} x) + B \lambda^{\frac{1}{2}} \sinh(\lambda^{\frac{1}{2}} x).
\]

Imposing \( y'(0) = 0 \) yields \( A \lambda^{\frac{1}{2}} = 0 \). Thus either \( A = 0 \) or \( \lambda = 0 \). Try the latter first. In that case \( y(x) = B \); imposing the other condition \( y(L) = 0 \) then forces \( B = 0 \), which produces the trivial solution \( y(x) \equiv 0 \). So instead we must have \( A = 0 \) and thus \( y(x) = B \cosh(\lambda^{\frac{1}{2}} x) \). Now imposing \( y(L) = 0 \) gives \( \cosh(\lambda^{\frac{1}{2}} L) = 0 \), which is solved by \( \lambda^{\frac{1}{2}} L = i(n + 1/2)\pi \) with \( n = 0, 1, 2, 3, \ldots \). Since \( \lambda^{\frac{1}{2}} \) is imaginary, the eigenfunction becomes

\[
y_n(x) = B_n \cos((n + 1/2)\pi x / L).
\]

By choosing \( B_n = \sqrt{2/L} \) the solutions are normalized. The final answer is thus

\[
\lambda = -\left(\frac{(n + 1/2)\pi}{L}\right)^2, \quad y_n(x) = \sqrt{\frac{2}{L}} \cos \left(\frac{(n + 1/2)\pi x}{L}\right), \quad n = 1, 2, 3, \ldots .
\]

It is easy to show (for instance by checking an integral table) that these are orthonormal:

\[
\langle y_n | y_m \rangle = \frac{2}{L} \int_0^L dx \sin \left(\frac{(n + 1/2)\pi x}{L}\right) \sin \left(\frac{(m + 1/2)\pi x}{L}\right) = \delta_{nm}.
\]

It would have been easier to solve this by realizing from the beginning that the solutions had to be sinusoidal rather than hyperbolic. That means \( \lambda \) must be negative. Thus we could write \( \lambda = -k^2 \). The ODE becomes

\[
\frac{d^2 y}{dx^2} = -k^2 y,
\]
which is solved by

\[ y(x) = A \sin(kx) + B \cos(kx). \]

The boundary condition \( y'(0) = 0 \) forces \( A = 0 \), so the eigenfunction becomes

\[ y(x) = B \cos(kx). \]

Now imposing the other boundary condition \( y(L) = 0 \) forces

\[ \cos(kL) = 0 \text{ or } kL = (n + 1/2)\pi. \]

Then the above solution follows easily.