Homework 13 Solution
PHZ 5156, Computational Physics
December 1, 2005

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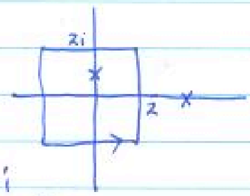
① $\oint_c \frac{dz}{(z-i)(z-3)}$



Only the pole at $z=i$ is inside C, so $\oint_c = 2\pi i \cdot (\text{residue at } z=i)$

$= 2\pi i \cdot \lim_{z\to i} \cancel{(z-i)} \frac{1}{\cancel{(z-i)}(z-3)} = \frac{2\pi i}{i-3} = \frac{2\pi}{1+3i}$

② $F(k) = \int_{-\infty}^{\infty} dx \frac{e^{ikx}}{(x+3i)^2}$ pole at $z=-3i$

$e^{ikz} = e^{ik(x+iy)} = e^{ikx} e^{-ky}$ $k>0$: close UHP, $k<0$: close LHP

$k>0$ $F(k) = 2\pi i \cdot (\text{sum of residues in UHP}) = 0$

$k<0$ $F(k) = -2\pi i\,(\text{sum of res in LHP})$

$f(z) = \frac{e^{ikz}}{(z+3i)^2}$ pole of order 2 at $z=-3i$

$\text{Res } f(-3i) = \frac{1}{(2-1)!} \frac{d}{dz} \cancel{(z+3i)^2} \frac{e^{ikz}}{\cancel{(z+3i)^2}} \Big|_{z=-3i} = ik e^{3k}$

$F(k) = -2\pi i\,(ike^{3k}) = 2\pi k e^{3k}$ $k<0$

so $F(k) = \begin{cases} 0, & k>0 \\ 2\pi k e^{3k}, & k<0 \end{cases}$

p.2

③ $f(x) = e^{-|x|}$

(a) $F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-|x|} e^{-ikx}$

$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{0} dx\, e^{x} e^{-ikx} + \int_{0}^{\infty} dx\, e^{-x} e^{-ikx} \right\}$

$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{0} dx\, e^{(1-ik)x} + \int_{0}^{\infty} dx\, e^{-(1+ik)x} \right\}$

$= \frac{1}{\sqrt{2\pi}} \left\{ \frac{e^{(1-ik)x}}{1-ik} \Big|_{-\infty}^{0} + \frac{e^{-(1+ik)x}}{-(1+ik)} \Big|_{0}^{\infty} \right\}$

$= \frac{1}{\sqrt{2\pi}} \left\{ \frac{1}{1-ik} + \frac{1}{1+ik} \right\} = \sqrt{\frac{2}{\pi}} \frac{1}{1+k^2}$

(b) $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, \sqrt{\frac{2}{\pi}} \frac{1}{1+k^2} e^{ikx} = \frac{1}{\pi} \int_{-\infty}^{\infty} dk \frac{e^{ikx}}{(k+i)(k-i)}$

put $k = k' + ik''$ $\quad e^{ikx} = e^{ik'x} e^{-k''x}$

$x>0$ close in UHP $\quad f(x) = 2\pi i\,(\text{res in UHP})$

pole at $k=i$ in UHP $\quad$ residue $= \frac{1}{\pi} \frac{e^{i i x}}{i+i} = \frac{e^{-x}}{2\pi i}$

$f(x) = 2\pi i \left(\frac{e^{-x}}{2\pi i} \right) = e^{-x} \quad x>0$

$x<0$ close in LHP $\quad f(x) = -2\pi i\,(\text{res in LHP})$

pole at $k=-i$ in LHP $\quad$ residue $= \frac{1}{\pi} \frac{e^{i(-i)x}}{-i-i} = -\frac{1}{2\pi i} e^{x}$

$f(x) = -2\pi i \left(-\frac{1}{2\pi i} e^{x} \right) = e^{x} \quad x<0$

so $\underline{f(x) = e^{-|x|}}$

~~note 2 this interesting thing: for x=0 can close in either the~~

p.3

(4) $f(x) = \dfrac{1}{(x^2+4)^2}$

(a) $F(k) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} dx \, \frac{e^{-ikx}}{(x^2+4)^2}$

put $z = x+iy$ $\quad e^{-ikz} = e^{-ik(x+iy)} = e^{-ikx} e^{ky}$

• $k>0$ close in LHP $\qquad F(k) = -2\pi i \,(\text{Res in LHP})$

$$\frac{1}{(z^2+4)^2} = \frac{1}{(z+2i)^2(z-2i)^2} \qquad \text{double pole}$$

Residue at $-2i$:

$$\lim_{z\to -2i} \frac{d}{dz} \cancel{(z+2i)^2} \frac{1}{\sqrt{2\pi}} \frac{e^{-ikz}}{\cancel{(z+2i)^2}(z-2i)^2}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{(z-2i)^{\cancel{2}}(-ik)e^{-ikz} - e^{-ikz}\,2\cancel{(z-2i)}}{(z-2i)^{\cancel{4}\,3}}\Bigg|_{-2i}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-2k} \frac{(-4i)(-ik)-2}{(-4i)^3} = -\frac{i}{64}\frac{1}{\sqrt{2\pi}} e^{-2k}(-)(2+4k)$$

$$= +\frac{i}{32}\frac{1}{\sqrt{2\pi}}(1+2k)e^{-2k}$$

$$F(k) = -2\pi i \,(\,\text{''}\,) = \frac{\sqrt{2\pi}}{32}(1+2k)e^{-2k} \qquad k>0$$

• $k<0$ close in UHP $\qquad F(k) = 2\pi i \,(\text{res in UHP})$

$$\text{Res at } 2i = \lim_{z\to 2i} \frac{d}{dz}\frac{1}{\sqrt{2\pi}}\frac{e^{-ikz}}{(z+2i)^2} = \frac{1}{\sqrt{2\pi}}\frac{(z+2i)(-ik)-2}{(z+2i)^3} e^{-ikz}\Bigg|_{z=2i}$$

$$= \frac{1}{\sqrt{2\pi}}\frac{4i(-ik)-2}{(4i)^3} e^{2k} = -\frac{i}{32}\frac{1}{\sqrt{2\pi}}(1-2k)e^{2k}$$

$$F(k) = 2\pi i \,(\,\text{''}\,) = \frac{\sqrt{2\pi}}{32}(1-2k)e^{2k} \qquad k<0$$

p.4

(4) (b) $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \; F(k)\, e^{ikx}$

$$= \frac{1}{\sqrt{2\pi}} \frac{\sqrt{2\pi}}{32} \left\{ \int_{-\infty}^{0} dk\, (1-2k)\, e^{2k} e^{ikx} + \int_{0}^{\infty} dk\, (1+2k)\, e^{-2k} e^{ikx} \right\}$$

$$= \frac{1}{32} \left\{ \int_{-\infty}^{0} dk\, (1-2k)\, e^{(2+ix)k} + \int_{0}^{\infty} dk\, (1+2k)\, e^{-(2-ix)k} \right\}$$

$$\int_0^\infty dk\, e^{-ak} = \frac{1}{a} \qquad \int_0^\infty dk\, k e^{-ak} = \frac{1}{a^2}$$

$$\int_{-\infty}^0 dk\, e^{ak} = \frac{1}{a} \qquad \int_{-\infty}^0 dk\, k e^{ak} = -\frac{1}{a^2}$$

$$f(x) = \frac{1}{32} \left\{ \frac{1}{2+ix} + 2\frac{1}{(2+ix)^2} + \frac{1}{2-ix} + 2\frac{1}{(2-ix)^2} \right\}$$

$$= \frac{1}{32} \left\{ \frac{2-ix+2+ix}{4+x^2} + 2\, \frac{(2+ix)^2 + (2-ix)^2}{(4+x^2)^2} \right\}$$

$$= \frac{1}{32} \left\{ \frac{4}{4+x^2} + 2\, \frac{8-2x^2}{(4+x^2)^2} \right\}$$

$$= \frac{1}{32}\, \frac{4(4+x^2) + 16 - 4x^2}{(4+x^2)^2} = \frac{1}{(4+x^2)^2}$$

p.5

(5) $f(x) = \begin{cases} x e^{-x} & x > 0 \\ 0 & x < 0 \end{cases}$

(a) $F(k) = \frac{1}{\sqrt{2\pi}} \int_0^\infty dx \; x e^{-x} e^{-ikx} = \frac{1}{\sqrt{2\pi}} \int_0^\infty dx \; x e^{-(1+ik)x}$

$F(k) = \frac{1}{\sqrt{2\pi}} \frac{1}{(1+ik)^2}$

(b) $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \; \frac{1}{\sqrt{2\pi}} \frac{1}{(1+ik)^2} e^{ikx}$

$\cancel{z = x + iy \quad e^{ikz} = e^{ikx} e^{-ky}}$

$k = k' + ik'' \quad e^{i(k' + ik'')x} = e^{ik'x} e^{-k''x}$

$x > 0$: close in UHP $\quad f(x) = 2\pi i \,(\text{res in UHP})$

pole at $k = i$ of 2d order $\quad \frac{1}{(1+ik)^2} = \frac{(-i)^2}{(-i+k)^2} = -\frac{1}{(k-i)^2}$

residue $= \lim_{k \to i} \cancel{(k-i)}\, \frac{d}{dk} (k-i)^2 \left(-\frac{1}{\sqrt{2\pi}}\right) \frac{e^{ikx}}{(k-i)^2} \frac{1}{\sqrt{2\pi}}$

$= -\frac{1}{2\pi} (ix) e^{ikx} \Big|_i = -\frac{i}{2\pi} x e^{-x}$

$f(x) = 2\pi i \,(\text{"}) = x e^{-x} \quad x > 0$ ✓

$x < 0$: close in LHP no poles $\Rightarrow f(x) = 0$ ✓