Homework 12 Solution
PHZ 5156, Computational Physics
November 30, 2005

PHZ 5156 HW12 11/30/05 p1

Lea Chapter 2

(20) $f(z) = \sin z^2$ put $z = z^2 - \pi + \pi$

$= \sin(z^2 - \pi + \pi) = -\sin(z^2 - \pi)$

near $z = \sqrt{\pi}$ this goes to $-\left[(z^2-\pi) - \frac{1}{6}(z^2-\pi)^3 + \cdots\right]$

which goes as $-(z-\sqrt{\pi})(z+\sqrt{\pi})$ + higher order

$= -2\sqrt{\pi}\,(z-\sqrt{\pi})$ + higher order

$\hookrightarrow$ so zero is of order 1

(21) (a) $z\cos z = z\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n+1}$ entire function $\Rightarrow R = \infty$

(b) Use $\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots$ Take integral

$\Rightarrow \ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{z^{n+1}}{n+1}$

singularity at $z = -1$ 

$\Rightarrow R = 1$

(c) $\frac{\sin z}{z}$ about $\frac{\pi}{2}$

$\sin z = \sin\left(z - \frac{\pi}{2} + \frac{\pi}{2}\right) = \cos\left(z - \frac{\pi}{2}\right) = 1 - \frac{1}{2}\left(z-\frac{\pi}{2}\right)^2 + \frac{1}{24}\left(z-\frac{\pi}{2}\right)^4 - \cdots$

$\frac{1}{z} = \frac{1}{z - \frac{\pi}{2} + \frac{\pi}{2}} = \frac{2}{\pi}\frac{1}{1 + \frac{2}{\pi}\left(z-\frac{\pi}{2}\right)} = \frac{2}{\pi}\left\{1 - \frac{2}{\pi}\left(z-\frac{\pi}{2}\right) + \left[\frac{2}{\pi}\left(z-\frac{\pi}{2}\right)\right]^2 - [\;]^3 \cdots\right\}$

$\frac{\sin z}{z} = \frac{2}{\pi}\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right)\left[1 - \frac{2}{\pi}x + \left(\frac{2}{\pi}x\right)^2 - \left(\frac{2}{\pi}x\right)^3 + \cdots\right]$ $x = z - \frac{\pi}{2}$

$= \frac{2}{\pi}\left\{1 - \frac{2}{\pi}x + \left[\left(\frac{2}{\pi}\right)^2 - \frac{1}{2}\right]x^2 + \left[-\left(\frac{2}{\pi}\right)^3 + \frac{1}{\pi}\right]x^3 + \cdots\right\}$

$\frac{\sin z}{z} = \frac{2}{\pi} - \left(\frac{2}{\pi}\right)^2\left(z-\frac{\pi}{2}\right) + \left[\left(\frac{2}{\pi}\right)^3 - \frac{1}{2}\left(\frac{2}{\pi}\right)\right]\left(z-\frac{\pi}{2}\right)^2 + \left[-\left(\frac{2}{\pi}\right)^4 + \frac{1}{2}\left(\frac{2}{\pi}\right)^2\right]\left(z-\frac{\pi}{2}\right)^3 + \cdots$

$R = \pi/2$ (singularity at $z = \pi/2$)

p2

21 (d) $\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} = \frac{1}{(z-2+1)(z-2+3)}$

$= \frac{1}{3} \cdot \frac{1}{1+\frac{1}{3}(z-2)} \cdot \frac{1}{1+(z-2)}$

$= \frac{1}{3} \sum_{m=0}^{\infty} (-1)^m \left[\frac{1}{3}(z-2)\right]^m \cdot \sum_{n=0}^{\infty} (-1)^n (z-2)^n$

$= \frac{1}{3} \sum_{m,n=0}^{\infty} \frac{(-1)^{m+n}}{3^m} (z-2)^{m+n}$ write as $\frac{1}{3} \sum_{k=0}^{\infty} a_k (z-2)^k$

$m+n=0$: $m=0, n=0$: $\frac{1}{1} = 1$

$m+n=1$: $(m=1\ n=0) + (m=0\ n=1)$

$\left(-\frac{1}{3}\right) + (-1) = -\frac{4}{3}$

$m+n=2$: $(m=2\ n=0) + (m=1\ n=1) + (m=0\ n=2)$

$\frac{1}{9} + \frac{1}{3} + 1 = \frac{13}{9}$

$m+n=k$: $\sum_{m=0}^{k} \frac{(-1)^k}{3^m} = (-1)^k \sum_{m=0}^{k} \frac{1}{3^m} = (-1)^k \frac{1-\left(\frac{1}{3}\right)^{k+1}}{1-1/3}$

$= (-1)^k \frac{3}{2}\left[1-\left(\frac{1}{3}\right)^{k+1}\right]$

$$\frac{1}{z^2-1} = \frac{1}{2}\sum_{k=0}^{\infty} (-1)^k \left[1-\left(\frac{1}{3}\right)^{k+1}\right](z-2)^k$$

poles at $z = \pm 1$

$\Rightarrow R = 1$

(22) (a) $\cos z = \cos(z-1+1) = \cos(z-1)\cos 1 - \sin(z-1)\sin 1$

$= \cos 1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(z-1)^{2n} - \sin 1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(z-1)^{2n+1}$

$\frac{\cos z}{z-1} = \cos 1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(z-1)^{2n-1} - \sin 1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(z-1)^{2n}$

$= \cancel{\sum_{k=-1}^{\infty} a_k (z-1)^k \text{ with } a_k}$

$= \frac{\cos 1}{z-1} - \sin 1 - \frac{\cos 1}{2}(z-1) + \frac{\sin 1}{6}(z-1)^2 + \cdots$

No other singularity $\Rightarrow$ $R=\infty$

(b) $\frac{\sin z^2}{z} = \frac{1}{z}\left(z^2 - \frac{1}{6}z^6 + \frac{1}{5!}z^{10} - \cdots\right) = z - \frac{z^5}{6} + \frac{z^9}{5!} - \cdots$

no other singularity $\Rightarrow$ $R=\infty$

(c) $e^z = e^{z-i\pi+i\pi} = e^{i\pi}e^{z-i\pi} = -e^{z-i\pi}$

$= -\sum_{n=0}^{\infty} \frac{(z-i\pi)^n}{n!}$ so $\frac{e^z}{z-i\pi} = -\sum_{n=0}^{\infty} \frac{(z-i\pi)^{n-1}}{n!}$

$= -\frac{1}{z-i\pi} - 1 - \frac{(z-i\pi)}{2} - \cdots$ no other singularity $\Rightarrow$ $R=\infty$

(d) $\ln z = \ln(z-1+1) = \ln[1-(1-z)] = \sum_{n=0}^{\infty} \frac{(1-z)^{n+1}}{n+1}$

$\frac{\ln z}{z-1} = -\frac{\ln z}{1-z} = -\sum_{n=0}^{\infty} \frac{(1-z)^n}{n+1} = -1 - \frac{1-z}{2} - \frac{(1-z)^2}{3} - \cdots$

singularity at $z=0$ $\Rightarrow$ $R=1$

p4

(23) (a) $e^{z} = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots$

$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \cdots$ for $|z| < 1$

$\frac{e^z}{1+z^2} = \left(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \cdots\right)\left(1 - z^2 + z^4 - \cdots\right)$

$= 1 + z - \frac{z^2}{2} - \frac{5z^3}{6} + \frac{13}{24} z^4 + \cdots \qquad |z| < 1$

for $|z| > 1$ can use $\frac{1}{1+z^2} = \frac{1}{z^2}\,\frac{1}{1+z^{-2}}$

$= \frac{1}{z^2}\left(1 - \frac{1}{z^2} + \frac{1}{z^4} - \frac{1}{z^6} + \cdots\right)$ but I won't finish – would need to multiply the two series & collect terms

(b) $\frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)} \qquad \frac{1}{z+i} = \frac{1}{z-i+2i} = \frac{1}{2i}\,\frac{1}{1-\frac{i}{2}(z-i)}$

$\frac{1}{z^2+1} = \frac{1}{z-i}\,\frac{1}{2i}\sum_{n=0}^{\infty}\left[\frac{i}{2}(z-i)\right]^n = -\sum_{n=0}^{\infty}\left(\frac{i}{2}\right)^{n+1}(z-i)^{n-1}$

$|z-i| < 1$

For $|z-i| > 1$, expand $\frac{1}{z+i} = \frac{1}{z-i+2i} = \frac{1}{z-i}\,\frac{1}{1+\frac{2i}{z-i}}$

$\frac{1}{z^2+1} = \frac{1}{z-i}\,\frac{1}{z-i}\sum_{n=0}^{\infty}\left(-\frac{2i}{z-i}\right)^n = \sum_{n=0}^{\infty}\frac{(-2i)^n}{(z-i)^{n+2}} \qquad |z-i| > 1$

(c) $\frac{z}{z^2-9} = \frac{z+3-3}{z^2-9} = \frac{z+3}{(z-3)(z+3)} - \frac{3}{(z-3)(z+3)} = \frac{1}{z-3} - \frac{3}{(z-3)(z+3)}$

$\frac{1}{z+3} = \frac{1}{z-3+6} = \frac{1}{6}\,\frac{1}{1+\frac{1}{6}(z-3)} = \frac{1}{6}\sum_{n=0}^{\infty}\left(-\frac{z-3}{6}\right)^n = \frac{1}{6} + \frac{1}{6}\sum_{n=1}^{\infty}\left(-\frac{z-3}{6}\right)^n$

$\frac{z}{z^2-9} = \frac{1}{z-3} - \frac{3}{z-3}\left\{\frac{1}{6} + \frac{1}{6}\sum_{n=1}^{\infty}\left(-\frac{z-3}{6}\right)^n\right\} = \frac{1}{2}\,\frac{1}{z-3} - \frac{1}{2}\sum_{n=1}^{\infty}\left(-\frac{1}{6}\right)^n (z-3)^{n-1}$

$= \frac{1}{2}\,\frac{1}{z-3} - \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{1}{6}\right)^{n+1}(z-3)^n \qquad |z-3| < 6$

I won't do $|z-3| > 6$

(d) $\frac{1}{9+z^2} = \frac{1}{9}\,\frac{1}{1+z^2/9} = \frac{1}{9}\sum_{n=0}^{\infty}\left(-\frac{z^2}{9}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{9^{n+1}} z^{2n} \qquad |z| < 3$

$\frac{1}{9+z^2} = \frac{1}{z^2}\,\frac{1}{1+9/z^2} = \frac{1}{z^2}\sum_{n=0}^{\infty}\left(-\frac{9}{z^2}\right)^n = \sum_{n=0}^{\infty}\frac{(-9)^n}{z^{2n+2}} \qquad |z| > 3$

p5

24

(a) $\frac{e^z}{z} - \sin\frac{1}{z}$ singular at $z=0$ and $\frac{1}{z} = n\pi$ $n = \pm1, \pm2, \ldots$

$\sin\frac{1}{z} = \frac{1}{z} - \frac{1}{6}\frac{1}{z^3} + \frac{1}{5!}\frac{1}{z^5} - \cdots$: all terms $\Rightarrow$ essential singularity

at $\frac{1}{z} = n\pi$, $n = \pm1, \pm2$, etc. :

put $\frac{1}{z} = \frac{1}{z} - n\pi + n\pi$: $\sin\left(\frac{1}{z} - n\pi + n\pi\right) = (-1)^n \sin\left(\frac{1}{z} - n\pi\right)$

$\cancel{\sin\frac{1}{z} = (-1)^n \sin\left(\frac{1-n\pi z}{z}\right)}$

$\sin\frac{1}{z} = (-1)^n \sin\left(\frac{1}{z} - n\pi\right) = (-1)^n \left[\left(\frac{1}{z} - n\pi\right) - \frac{1}{6}(\;)^3 + \cdots\right)$

$\frac{1}{z} - n\pi = \frac{1}{z - \frac{1}{n\pi} + \frac{1}{n\pi}} - n\pi = n\pi\left[\frac{1}{1 + n\pi\left(z - \frac{1}{n\pi}\right)} - 1\right]$

$= n\pi\left[\cancel{1} - n\pi\left(z - \frac{1}{n\pi}\right) + (\;)^2 \cdots \cancel{-1}\right]$

$= -n\pi\left\{\left[n\pi\left(z - \frac{1}{n\pi}\right)\right] - [\;]^2 + \cdots\right\}$

so $\sin\frac{1}{z} = (-1)^n(-n\pi)\,n\pi\left(z - \frac{1}{n\pi}\right) + O\left(z - \frac{1}{n\pi}\right)^2 + \cdots$

$\Rightarrow$ pole of order 1 at $z = \frac{1}{n\pi}$ $n = \pm1, \pm2, \ldots$

(b) $\frac{\cos z}{z} - \frac{\sin z}{z^2}$ singular at $z = 0$

$\frac{1}{z}\left(1 - \frac{1}{2}z^2 + \frac{z^4}{24} - \cdots\right) - \frac{1}{z^2}\left(z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \cdots\right)$

$= \cancel{\frac{1}{z}} - \frac{z}{2} + \frac{z^3}{24} - \cdots - \cancel{\frac{1}{z}} + \frac{z}{6} - \frac{z^3}{120} - \cdots$

$= -\frac{z}{3} + \cdots$ $\Rightarrow$ removable singularity

(c) $\frac{\tanh z}{z} \to \frac{z}{z} = 1$ as $z \to 0$ so removable sing at $z=0$

oops: also singularities at $z = i\left(n + \frac{1}{2}\right)\pi$ (simple poles)

(d) $\ln(1+z^2)$ singular at $z = \pm i$: branch points

(27) (a) residue of $\frac{z-2}{z^2-1}$ at $z=1$

simple pole: $\text{res} = \lim_{z\to 1} (z-1)\frac{z-2}{(z-1)(z+1)} = -\frac{1}{2}$

(b) $e^{\frac{1}{z}-1} = e^{-1}e^{\frac{1}{z}} = e^{-1}\left(1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \cdots\right)$

residue $= \frac{1}{e}$ at $z=0$

(c) $\frac{\sin z}{z^2} \to \frac{1}{z}$ at $z=0$ so residue at $z=0$ is 1

(d) $\frac{\cos z}{\frac{1}{2}-\sin z}$ at $z=\pi/6$ $\quad \sin\frac{\pi}{6} = \frac{1}{2}$ so pole at $z=\frac{\pi}{6}$

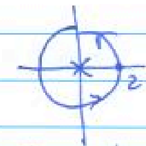
$$\sin z = \sin\left(z-\frac{\pi}{6}+\frac{\pi}{6}\right) = \sin\left(z-\frac{\pi}{6}\right)\cos\frac{\pi}{6} + \cos\left(z-\frac{\pi}{6}\right)\sin\frac{\pi}{6}$$
$$= \frac{\sqrt{3}}{2}\sin\left(z-\frac{\pi}{6}\right) + \frac{1}{2}\cos\left(z-\frac{\pi}{6}\right)$$

for z near $\frac{\pi}{6}$, $\sin z \approx \frac{1}{2} + \frac{\sqrt{3}}{2}\left(z-\frac{\pi}{6}\right) + O\left(z-\frac{\pi}{6}\right)^2$

$\frac{1}{2} - \sin z = -\frac{\sqrt{3}}{2}\left(z-\frac{\pi}{6}\right) + O\left(z-\frac{\pi}{6}\right)^2$

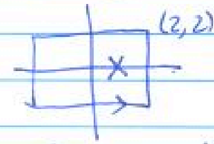
$$\frac{\cos z}{\frac{1}{2}-\sin z} \approx \frac{\sqrt{3}/2}{-\frac{\sqrt{3}}{2}\left(z-\frac{\pi}{6}\right)} = -\frac{1}{z-\frac{\pi}{6}} \quad \text{so Res} = -1$$

p7

(28) (a) $\oint_c \frac{\cos z}{z}\,dz$  simple pole at $z=0$

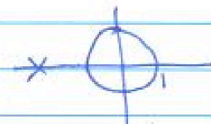
$\oint_c = 2\pi i \cdot (\text{residue at } z=0)$

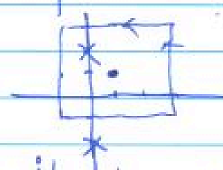
$= 2\pi i \cdot \lim_{z\to 0} z\,\frac{\cos z}{z} = 2\pi i$

(b) $\oint_c \frac{\sinh z}{z-1}\,dz$ 

simple pole at $z=1$

$\oint_c = 2\pi i \cdot (\text{res at } z=1) = 2\pi i \cdot \lim_{z\to 1} (z-1)\,\frac{\sinh z}{z-1} = 2\pi i \sinh 1$

(c) $\oint_c \frac{z-1}{z+2}\,dz$  no poles inside $\Rightarrow \oint_c = 0$

(d) $\oint_c \frac{z}{4z^2+1}\,dz$  poles at $z = \pm\frac{i}{2}$

Only pole at $z = i/2$ is inside

$\oint = 2\pi i \cdot (\text{res at } z=\frac{i}{2}) = 2\pi i \lim_{z\to \frac{i}{2}} \cancel{(z-\frac{i}{2})}\,\frac{z}{4\cancel{(z-\frac{i}{2})}(z+\frac{i}{2})}$

$= 2\pi i\,\frac{i/2}{4i} = \frac{\pi i}{4}$

p 8

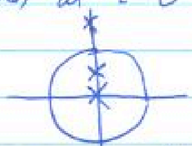
(29) (a) $\int_0^{2\pi} \frac{1+\cos\theta}{2-\sin\theta}\, d\theta$ $\quad z = e^{i\theta} \quad dz = ie^{i\theta}d\theta = iz\,d\theta \quad d\theta = -i\frac{dz}{z}$

$\sin\theta = \frac{1}{2i}(z-z^{-1}) \quad \cos\theta = \frac{1}{2}(z+z^{-1})$

$-i\oint_c \frac{1+\frac{1}{2}(z+z^{-1})}{2-\frac{1}{2i}(z-z^{-1})}\frac{dz}{z}$ multiply by $\frac{2iz}{2iz}$

$= -i\oint_c \frac{i(2z+z^2+1)}{4iz-z^2+1}\frac{dz}{z} = (-)\oint \frac{dz}{z}\frac{2z+z^2+1}{z^2-4iz-1}$

poles at $z=0$ and $z^2-4iz-1=0 \quad z=\frac{1}{2}[4i\pm\sqrt{-16+4}]$



$z = \frac{1}{2}(4i \pm i\sqrt{12}) = i(2\pm\sqrt{3})$

$z=0$ & $z=i(2-\sqrt{3})$ inside

res $f(0) = \lim_{z\to 0} \cancel{z}\frac{1}{\cancel{z}}\frac{2z+z^2+1}{z^2-4iz-1} = -1$

res $f(i(2-\sqrt{3})) = \lim_{z\to i(2-\sqrt{3})} (z-z_0)\frac{1}{z}\frac{2z+z^2+1}{[z-i(2-\sqrt{3})][z-i(2+\sqrt{3})]}$

$= \frac{2i(2-\sqrt{3})-(2-\sqrt{3})^2+1}{i(2-\sqrt{3})[i(2-\sqrt{3})-i(2+\sqrt{3})]} = \frac{2i(2-\sqrt{3})-(2-\sqrt{3})^2+1}{(-1)(2-\sqrt{3})(-2\sqrt{3})}$

$= \frac{2i(2-\sqrt{3})-(2-\sqrt{3})^2+1}{2\sqrt{3}(2-\sqrt{3})} = \frac{[1+i(2-\sqrt{3})]^2}{2\sqrt{3}(2-\sqrt{3})}$

$\oint_c = (-)2\pi i\left(-1+\frac{[1+i(2-\sqrt{3})]^2}{2\sqrt{3}(2-\sqrt{3})}\right) = -2\pi i\frac{\cancel{2}i}{\cancel{2}\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$

p 9

(b) $\int_0^{\pi} \frac{\sin^2\theta}{1+\cos^2\theta}\,d\theta$ use $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$= \int_0^{\pi} \frac{\cancel{\frac{1}{2}}(1-\cos 2\theta)}{\cancel{\frac{1}{2}}(3+\cos 2\theta)}\,d\theta$ $\quad \theta' = 2\theta \quad d\theta' = 2d\theta$

$= \frac{1}{2}\int_0^{2\pi} \frac{1-\cos\theta'}{3+\cos\theta'}\,d\theta'$ same substitution

$= -\frac{i}{2}\oint_C \frac{dz}{z}\,\frac{1-\frac{1}{2}(z+z^{-1})}{3+\frac{1}{2}(z+z^{-1})} = -\frac{i}{2}\oint \frac{dz}{z}\,\frac{2z-z^2-1}{6z+z^2+1}$

poles at $z=0$ and $z^2+6z+1=0$

$z = \frac{1}{2}\left(-6 \pm \sqrt{36-4}\right) = \frac{1}{2}\left(-6 \pm \sqrt{32}\right) = -3 \pm \sqrt{8}$

$-3+\sqrt{8}$ is inside

$S = \underbrace{2\pi i\left(-\frac{i}{2}\right)}_{\pi}\left[(\text{res at } 0) + (\text{res at } -3+\sqrt{8})\right]$

res at $0 = \lim_{z\to 0} z \frac{1}{z}\frac{2z-z^2-1}{6z+z^2+1} = -1$ $\quad$ num $= -(z^2-2z+1)$ $= -(z-1)^2$

res at $-3+\sqrt{8} = \lim_{z\to\sqrt{8}-3} \cancel{(z-(\sqrt{8}-3))}\frac{1}{z}\frac{-(z-1)^2}{\cancel{[z-(\sqrt{8}-3)]}[z-(-\sqrt{8}-3)]}$

$= -\frac{1}{\sqrt{8}-3}\,\frac{(\sqrt{8}-3-1)^2}{2\sqrt{8}} = \frac{1}{3-\sqrt{8}}\,\frac{1}{2\sqrt{8}}\left(4-\sqrt{8}\right)^2$

$S = \pi\left[-1 + \frac{(4-\sqrt{8})^2}{2\sqrt{8}(3-\sqrt{8})}\right] = \underline{\pi(\sqrt{2}-1)}$

p10

(c) $\int_0^{2\pi} \frac{d\theta}{1+\sin^2\theta}$ use $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$$= \int_0^{2\pi} \frac{d\theta}{\frac{1}{2}(3-\cos 2\theta)} = 2\int_0^{2\pi} \frac{d\theta}{3-\cos 2\theta} = \int_0^{4\pi} \frac{d\theta}{3-\cos\theta}$$

$$= 2\int_0^{2\pi} \frac{d\theta}{3-\cos\theta} = -2i \oint \frac{dz}{z} \frac{1}{3-\frac{1}{2}(z+z^{-1})} = -4i \oint_c \frac{dz}{6z-z^2-1}$$

$$= 4i \oint_c \frac{dz}{z^2-6z+1}$$

poles $z = \frac{1}{2}\left(6 \pm \sqrt{36-4}\right) = \frac{1}{2}\left(6 \pm \sqrt{32}\right) = 3 \pm \sqrt{8}$

only $3-\sqrt{8}$ inside

$$\oint_c = 2\pi i\,(4i) \lim_{z \to 3-\sqrt{8}} \cancel{[z-(3-\sqrt{8})]} \frac{1}{\cancel{[z-(3-\sqrt{8})]}\,[z-(3+\sqrt{8})]}$$

$$= -8\pi \frac{1}{-2\sqrt{8}} = \frac{\sqrt{8}\,\pi}{2} = \underline{\pi\sqrt{2}}$$

p 11

(d) $\int_0^{\pi} \sin^{2n}\theta\, d\theta = \frac{1}{2}\int_0^{2\pi} \sin^{2n}\theta\, d\theta$

$= -\frac{i}{2}\oint_c \frac{dz}{z}\left(\frac{z-z^{-1}}{2i}\right)^{2n} = \frac{1}{(2i)^{2n+1}}\oint_c \frac{dz}{z}\left(z-z^{-1}\right)^{2n}$

Use $(a+b)^n = \sum_{k=0}^{n}\binom{n}{k} a^k b^{n-k}$

$(z-z^{-1})^{2n} = \sum_{k=0}^{2n}\binom{2n}{k} z^k (-z^{-1})^{2n-k}$

$= \sum_{k=0}^{2n}\binom{2n}{k} z^k z^{-2n+k} (-1)^k$

$= \sum_{k=0}^{2n}\binom{2n}{k}(-1)^k z^{2k-2n}$

$f(z) = \frac{1}{z}(z-z^{-1})^{2n} = \sum_{k=0}^{2n}(-1)^k\binom{2n}{k} z^{2k-2n-1} \quad (*)$

Could have looked at $\frac{1}{z}(z-z^{-1})^{2n} = \frac{1}{z\cdot z^{2n}}(z^2-1)^{2n}$
see have pole only at $z=0$

Look at (*): residue comes from term with $k=n$,
which gives power z^{-1}: res $= (-1)^n\binom{2n}{n}$

So $\int = 2\pi i \frac{1}{(2i)^{2n+1}}(-1)^n\binom{2n}{n} = \frac{\pi}{(2i)^{2n}}(-1)^n\binom{2n}{n}$

$\int = \frac{\pi}{4^n}\binom{2n}{n}$