Homework 11 Solution
PHZ 5156, Computational Physics
November 30, 2005

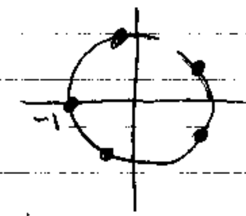
$\boxed{\text{HW 11}}$ p 1

Lea Ch 2

7 (a) $z^5 = -1$

$z = (-1)^{1/5}$ write $-1 = e^{i(2n+1)\pi}$ $n = 0,1,2,3,4$

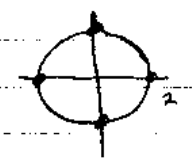
$z = \left[e^{i(2n+1)\pi}\right]^{1/5} = e^{i(2n+1)\frac{\pi}{5}} = e^{i\frac{\pi}{5}}, e^{i\frac{3\pi}{5}}, \underbrace{e^{i\pi}}_{-1}, e^{i\frac{7\pi}{5}}, e^{i\frac{9\pi}{5}}$



(b) $z = 16^{\frac{1}{4}}$ write $16 = 16e^{i2n\pi}$ $n = 0,1,2,3$

$= \left(16e^{i2n\pi}\right)^{\frac{1}{4}} = 2e^{\frac{in\pi}{2}} = 2, 2e^{i\frac{\pi}{2}}, 2e^{i\pi}, 2e^{i\frac{3\pi}{2}}$

$= 2, 2i, -2, -2i$



8 (a) $\cos z = 100$ write $z = x+iy$

$\cos z = \cos(x+iy) = \cos x \cos iy - \sin x \sin iy$

$= \cos x \cosh y - i \sin x \sinh y = 100$

$=$ real, so either $\sin x = 0$ or $\sinh y = 0$

- if $\sinh y = 0$, $y = 0$ & so $\cos z = \cos x$ ∴ can never be 100

- so must have $\sin x = 0 \Rightarrow x = n\pi$ $n = 0, \pm 1, \pm 2, \ldots$

$\cos z = \cos n\pi \cosh y = (-1)^n \cosh y = 100$

to be positive, need $n =$ even $= 2m$

then $\cosh y = 100 \Rightarrow y = \pm\cosh^{-1} 100$ $x = 2m\pi$

So $\underline{z = 2m\pi \pm i\cosh^{-1}(100)}$ $m = 0, \pm 1, \ldots$

$e^{iz} + e^{-iz} = 200$ $u = e^{iz}$ $u^2 - 200u + 1 = 0$ $u = \frac{1}{2}(200 \pm \sqrt{39996})$

$u = 100 \pm 3\sqrt{1111}$ $iz = \ln\left[u\, e^{i2m\pi}\right] = i2m\pi + \ln(100 \pm 3\sqrt{1111})$ ✓

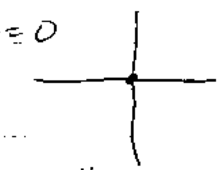
$z = 2m\pi + \ln(100 \pm 3\sqrt{1111}) = 2m\pi \pm i\ln(100 + 3\sqrt{1111})$

HW ~~8~~ p2

(10) $z = \ln(-5)$ write $-5 = 5e^{i(2n+1)\pi}$ $n = 0, \pm 1, \dots$

$z = \ln(5e^{i(2n+1)\pi}) = \ln 5 + i(2n+1)\pi$

(11) $w = z^{-\frac{1}{2}} = \frac{1}{z^{1/2}}$
- branch point at $z = 0$
- branch cut
- two branches



~~write $z = x+iy$, $w = u+iv$: $u+iv = \frac{1}{(x+iy)^{1/2}} = \frac{1}{(x+iy)^{1/2}}\left(\frac{x-iy}{x-iy}\right)^{1/2} = \frac{(x-iy)^{1/2}}{x^2 \dots}$~~

write $z = re^{i\theta}$ $w = (re^{i\theta})^{-\frac{1}{2}} = r^{-\frac{1}{2}}e^{-i\frac{1}{2}\theta}$

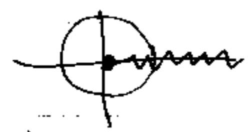
can replace θ by $\theta + 2n\pi$ to get all branches

$w = r^{-\frac{1}{2}} e^{-i\frac{1}{2}(\theta + 2n\pi)} = e^{-in\pi} r^{-\frac{1}{2}} e^{-i\frac{1}{2}\theta} = (-1)^n r^{-\frac{1}{2}} e^{-i\frac{1}{2}\theta}$

$w = \pm r^{-\frac{1}{2}} e^{-i\frac{1}{2}\theta}$ 2 choices $\Rightarrow$ 2 branches

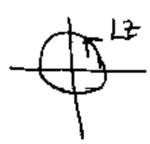
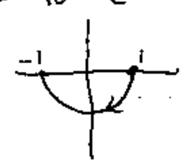
$u + iv = \pm r^{-\frac{1}{2}}(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2})$ ~~$= \pm\{r^{\frac{1}{2}}\cos\frac{\theta}{2} - r\sin\frac{\theta}{2}\}$~~

so $u = \pm r^{-\frac{1}{2}}\cos\frac{\theta}{2}$
$v = \mp r^{-\frac{1}{2}}\sin\frac{\theta}{2}$
upper / lower = { branch 1 / branch 2

Use this cut:  + first branch

As circle unit circle, $r = 1$ + θ goes 0 to 2π

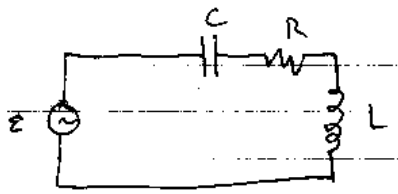
$w = e^{-i\frac{1}{2}\theta}$ goes 1 to $e^{-i\pi}$ clockwise

 maps to 

next branch maps to 

HW 0 p 3

(14)



$\varepsilon = \varepsilon_0 \cos\omega t$

(a) $\varepsilon - V_C - V_R - V_L = 0$

$V_R = IR$

$V_C = \frac{Q}{C} = \frac{1}{C}\int I\,dt$

$V_L = L\dot{I}$

(b) write $\varepsilon = \varepsilon_0 e^{i\omega t}$, $I = I_0 e^{i\omega t}$, & take real part at the end

$V_R = I_0 R e^{i\omega t}$

$V_C = \frac{1}{i\omega C} I_0 e^{i\omega t}$

$V_L = i\omega L I_0 e^{i\omega t}$

$\left[\varepsilon_0 - I_0\left(R + i\omega L - \frac{i}{\omega C}\right)\right] e^{i\omega t} = 0$

put $\underline{Z = R + i\left(\omega L - \frac{1}{\omega C}\right)}$

then $\varepsilon_0 = I_0 Z \Rightarrow I_0 = \frac{\varepsilon_0}{Z}$

so

(c) $I = \frac{\varepsilon_0}{Z} e^{i\omega t}$

write $Z = |Z| e^{i\phi}$

$|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$

$\tan\phi = \frac{\omega L - 1/\omega C}{R}$

$I = \frac{\varepsilon_0}{|Z|} e^{i(\omega t - \phi)}$ really $\boxed{\begin{aligned} I &= \frac{\varepsilon_0}{|Z|}\cos(\omega t - \phi) \\ I_0 &= \frac{\varepsilon_0}{|Z|} = \frac{\varepsilon_0}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} \\ \phi &= \tan^{-1}\left(\frac{1}{R}(\omega L - 1/\omega C)\right) \end{aligned}}$

(d) $P(t) = \varepsilon(t) I(t)$

$= \frac{\varepsilon_0^2}{|Z|}\cos\omega t \cos(\omega t - \phi) = \frac{\varepsilon_0^2}{|Z|}\left[\underbrace{\cos^2\omega t}_{\text{average } \frac{1}{2}} \cos\phi + \underbrace{\cos\omega t \sin\omega t}_{\text{average } = 0} \sin\phi\right]$

$= \frac{1}{2}\frac{\varepsilon_0^2}{|Z|}\cos\phi = \frac{1}{2}\mathrm{Re}\left(\varepsilon_0 e^{i\omega t} \frac{\varepsilon_0}{|Z|} e^{-i(\omega t - \phi)}\right)$ ✓

Hw # p4

(16) (a) $f = z^2 \sin z \qquad z = x+iy$

$= (x+iy)^2 \sin(x+iy)$

$= (x^2-y^2+2ixy)(\sin x \cosh y + i \cos x \sinh y)$

$u = (x^2-y^2)\sin x \cosh y - 2xy \cos x \sinh y$

$v = (x^2-y^2)\cos x \sinh y + 2xy \sin x \cosh y$

$\frac{\partial u}{\partial x} = 2x \sin x \cosh y + (x^2-y^2)\cos x \cosh y - 2y \sinh y(\cos x - x \sin x)$

$\frac{\partial v}{\partial y} = -2y \cos x \sinh y + (x^2-y^2)\cos x \cosh y + 2x \sin x(\cosh y + y \sinh y)$

equal ✓

$\frac{\partial v}{\partial x} = 2x \cos x \sinh y - (x^2-y^2)\sin x \sinh y + 2y \cosh y(\sin x + x \cos x)$

$\frac{\partial u}{\partial y} = -2y \sin x \cosh y + (x^2-y^2)\sin x \sinh y - 2x \cos x(\sinh y + y \cosh y)$

opposite ✓

$\frac{df}{dz} = 2z \sin z + z^2 \cos z$

(b) $f = \frac{1}{1+z} = \frac{1}{1+x+iy} \cdot \frac{1+x-iy}{1+x-iy} = \frac{1+x-iy}{(1+x)^2+y^2}$

$u = \frac{1+x}{(1+x)^2+y^2} \qquad v = -\frac{y}{(1+x)^2+y^2}$

$\frac{\partial u}{\partial x} = \frac{(1+x)^2+y^2-(1+x)2(1+x)}{den^2} = \frac{-(1+x)^2+y^2}{den^2}$

$\frac{\partial v}{\partial y} = -\frac{(1+x)^2+y^2-y\,2y}{den^2} = \frac{-(1+x)^2+y^2}{den^2}$ } equal

$\frac{\partial v}{\partial x} = -\frac{y\,2(1+x)}{den^2}$

$\frac{\partial u}{\partial y} = \frac{(1+x)2y}{den^2}$ } opposite

$\frac{df}{dz} = -\frac{1}{(1+z)^2}$

HW p5

① 

$z = 2e^{i\theta}$ $\quad dz = 2e^{i\theta}\, i\, d\theta = z\, i\, d\theta$

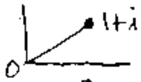
θ goes 0 to 2π

(a) $\oint \frac{dz}{z^2} = \int_0^{2\pi} \frac{2ie^{i\theta}d\theta}{4e^{i2\theta}} = \frac{i}{2}\int_0^{2\pi} e^{-i\theta}d\theta = 0$

(b) $\oint \frac{dz}{z} = \int_0^{2\pi} \frac{z\, i\, d\theta}{z} = \int_0^{2\pi} i\, d\theta = 2\pi i$

(c) $\oint dz = \int_0^{2\pi} 2ie^{i\theta} d\theta = 0$

(d) $\oint z\, dz = \int_0^{2\pi} 4ie^{2i\theta} d\theta = 0$

② 

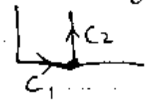
(a) $\int_c \cos z\, dz = \sin z \Big|_0^{1+i} = \sin(1+i)$

$= \sin 1 \cos i + \cos 1 \sin i = \sin 1 \cosh 1 + i \cos 1 \sinh 1$

(b) $z = x+iy = (1+i)x \qquad dz = (1+i)dx$

$\cos z = \cos(1+i)x$

$\int_c = \int_0^1 \cos((1+i)x) \cdot (1+i)dx = (1+i)\, \frac{\sin((1+i)x)}{1+i}\Big|_0^1 = \sin(1+i)$

(c) 

on C_1 $\quad z = x \qquad \int_{c_1} = \int_0^1 \cos x\, dx = \sin x\Big|_0^1 = \sin 1$

on C_2 $\quad z = 1+iy \quad dz = i\,dy$

$\int_{c_2} = \int_0 \cos(1+iy)\, i\, dy$

$= i \int_0^1 dy\, [\cos 1 \cosh y - i \sin 1 \sinh y)\, dy$

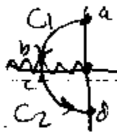
$= i\,[\cos 1 \sinh y - i \sin 1 \cosh y]\Big|_0^1$

$= i\,[\cos 1 \sinh 1 - i \sin 1 \cosh 1 + i \sin 1]$

$= \sin 1 \cosh 1 + i \cos 1 \sinh 1 - \sin 1$

$\int_{c_1} + \int_{c_2} = \sin 1 \cosh 1 + i \cos 1 \sinh 1$ ✓

HW p6

(3)  $C = C_1$ then C_2

$$\int_C = \int_{C_1} + \int_{C_2}$$

$$\int_{C_1} = \int_{C_1} z^{\frac{1}{2}}\,dz = \frac{2}{3} z^{\frac{3}{2}}\Big|_a^b \qquad a = e^{i\frac{\pi}{2}},\quad b = e^{i\pi}$$

$$= \frac{2}{3}\left[(e^{i\pi})^{\frac{3}{2}} - (e^{i\frac{\pi}{2}})^{\frac{3}{2}}\right] = \frac{2}{3}\left[e^{i\frac{3\pi}{2}} - e^{i\frac{3\pi}{4}}\right)$$

$$\int_{C_2} = \frac{2}{3} z^{\frac{3}{2}}\Big|_c^d \qquad c = e^{-i\pi},\quad d = e^{-i\frac{\pi}{2}}$$

$$= \frac{2}{3}\left[(e^{-i\frac{\pi}{2}})^{\frac{3}{2}} - (e^{-i\pi})^{\frac{3}{2}}\right] = \frac{2}{3}\left[e^{-i\frac{3\pi}{4}} - e^{-i\frac{3\pi}{2}}\right]$$

$$\int_{C_1+C_2} = \frac{2}{3}\left[e^{i\frac{3\pi}{2}} - e^{i\frac{3\pi}{4}} + e^{-i\frac{3\pi}{4}} - e^{-i\frac{3\pi}{2}}\right]$$

$$= \frac{2}{3}(2i)\left(\sin\frac{3\pi}{2} - \sin\frac{3\pi}{4}\right)$$

$$= \frac{4i}{3}\left(-1 - \frac{\sqrt{2}}{2}\right)$$

$$\int_C = -\frac{4i}{3}\left(1 + \frac{\sqrt{2}}{2}\right)$$