Homework 10 Solution
PHZ 5156, Computational Physics
November 30, 2005

Problem 1

Using separation of variables gives, for the boundary condition f(x,1)=x(1-x):

$$f(x,y) = \sum_{n=1}^{\infty} a_n \sin(k_n x)\sinh(k_n y)$$

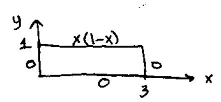
with $k_n = n\pi/3$ and

$$a_n = \frac{2}{3\sinh k_n}\frac{1}{k_n^3}\left[2 + \left(6k_n^2 - 2\right)(-1)^n\right]$$

See below for the derivation. Using the alternative boundary condition f(x,1)=x(3-x) gives a similar result with

$$a_n = \frac{2}{3\sinh k_n}\frac{2}{k_n^3}\left[1 - (-1)^n\right] = \begin{cases} \dfrac{8}{3k_n^3 \sinh k_n}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

(1)



$\nabla^2 f(x,y) = 0 \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$

Try $f(x,y) = g(x)h(y)$: $g''h + gh'' = 0$

Divide by $g(x)h(y)$: $\underbrace{\frac{g''(x)}{g(x)}}_{\lambda} + \underbrace{\frac{h''(y)}{h(y)}}_{-\lambda} = 0$

$g''(x) - \lambda g(x) = 0 \qquad h''(y) + \lambda h(y) = 0$

Need oscillating in x direction $\Rightarrow \lambda < 0$ so put $\lambda = -k^2$

$g''(x) + k^2 g(x) = 0 \qquad h''(y) - k^2 h(y) = 0$

$g(x) = A\cos kx + B\sin kx \qquad h(y) = C\cosh ky + D\sinh ky$

$g(0) = 0 = A \qquad h(0) = 0 = C$

so $g(x) = B\sin kx \qquad$ so $h(y) = \sinh k_n y$

$g(3) = 0 = \sin 3k$

so $k = k_n = \frac{n\pi}{3}$ $n = 1, 2, 3, \ldots$

Thus $f(x,y) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{3} \sinh\frac{n\pi y}{3}$

Apply inhomo BC: $f(x,1) = x(1-x) = \sum_{n=1}^{\infty}\left(a_n \sinh\frac{n\pi}{3}\right)\sin\frac{n\pi x}{3}$

multiply by $\sin\frac{m\pi x}{3}$ & $\int_0^3 dx$: $\int_0^3 dx\, x(1-x)\sin\frac{m\pi x}{3} = a_m \sinh\frac{m\pi}{3} \cdot \frac{3}{2}$

$a_n = \frac{2}{3\sinh(n\pi/3)}\int_0^3 dx\, x(1-x)\sin\frac{n\pi x}{3} = \frac{2}{3\sinh\frac{n\pi}{3}}\left\{\frac{1}{k^3}\left\{2 + (6k_n^2 - 2)(-1)^n\right\}\right\}$

$a_n = \frac{2}{3\sinh k_n}\frac{1}{k_n^3}\left[2 + (6k_n^2 - 2)(-1)^n\right]$

Problem 2

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#Homework 10
#PHZ 5156, Computational Physics, November 29, 2005

from scipy import *
import Gnuplot,Gnuplot.funcutils

nx = 31
h = 3./(nx-1.)
ny = int(1./h + 1 + 0.0000001)
print "max x,y=",(nx-1)*h,(ny-1)*h

f = zeros((nx,ny),Float)
x = h*arange(nx)
f[:,-1]=x*(1.-x)
f[-1,-1] = 0.          #I decided to make the ambiguous corner value zero.

nsweeps = 1000
tol = 0.00001
r = 0.5 * (cos(pi/nx)+cos(pi/ny))
w = 2./(1.+sqrt(1-r**2))

for sweep in arange(nsweeps):
    maxchange = 0.
    for i in arange(1,nx-1):
        for j in arange(1,ny-1):
            avg = 0.25*(f[i+1,j]+f[i-1,j]+f[i,j+1]+f[i,j-1])
            diff = avg - f[i,j]
            maxchange = max(maxchange,diff)
            f[i,j] = (1.-w)*f[i,j] + w*avg
    if  maxchange<tol:
        print "Finished after",sweep,"sweeps"
        print "maxchange",maxchange,"is less than tolerance",tol
        break
if sweep==nsweeps-1: print "Did not converge"

fanal = zeros((nx,ny),Float)
term = zeros((nx,ny),Float)

for n in arange(1,200):
    k = n*pi/3.
    a = ( 2. + (6.*k**2 - 2.)*((-1.)**n) ) / (k**3)
    a = 2. * a / (3. * sinh(k))
    for i in arange(nx):
        x = i*h
        for j in arange(ny):
            y = j*h
            term[i,j] = sin(n*pi*x/3.)*sinh(n*pi*y/3.)
    term = a*term
    fanal = fanal+term
```

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f1=open('/Users/michaeljohnson/physics/classes/5156/hw10/fnumeric.txt','w')
f2=open('/Users/michaeljohnson/physics/classes/5156/hw10/fanalytic.txt','w')
for i in arange(nx):
	for j in arange(ny):
		f1.write('%10.5f' % f[i,j])
		f2.write('%10.5f' % fanal[i,j])
	f1.write("\n")
	f2.write("\n")
f1.close()
f2.close()

#check how close the numeric and analytic are
maxdiff = max(max(abs(f-fanal)))
print "Maximum difference at any grid point is",maxdiff
```
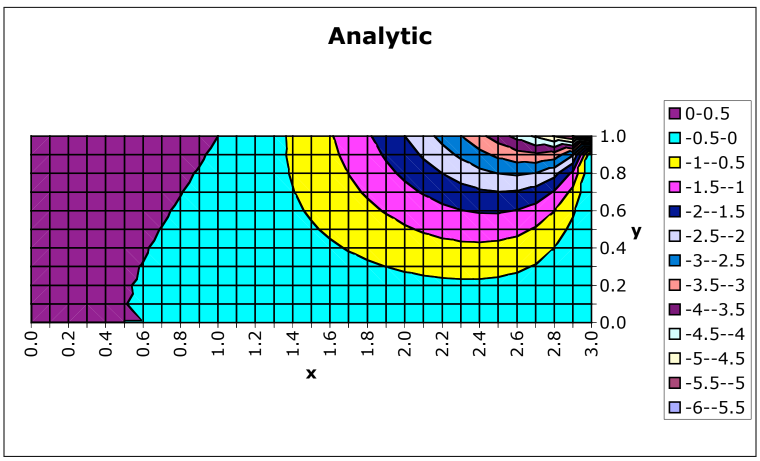
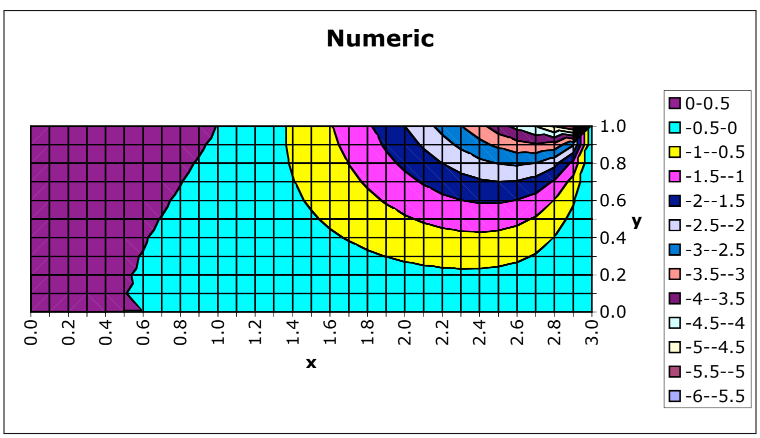

Output:
max x,y= 3.0 1.0
Finished after 42 sweeps
maxchange 9.96334339478e-06 is less than tolerance 1e-05
Maximum difference at any grid point is 0.074980847925





I output the data to a file and made contour plots using Excel. Obviously they look very similar.

Moreover, my test found that the numerical and analytical results never differed by more than 0.075 at any grid point.