INTERMEDIATE PHYSICS LAB MANUAL

UNIVERSITY OF CENTRAL FLORIDA

MANUEL MORALES

CONTENTS

1 CHARGE-TO-MASS RATIO

Given a current *source point* $\mathbf{r}_0$ and a *measuring point* $\mathbf{r}_p$, the *separation vector* is given by $\mathbf{r} = \mathbf{r}_p - \mathbf{r}_0$.

Definition 1.1. The Biot-Savart Law:

The magnetic field of a *steady* line current I is given by the *Biot-Savart law*:

$$\boxed{\mathbf{B(r)} = \frac{\mu_0}{4\pi} I \int \frac{\mathbf{dl'} \times \hat{\mathbf{r}}}{|r|^2}} \tag{1}$$

where the constant μ_0 is called the *permeability of free space* and $\mathbf{dl'}$ is a vector in the direction of current flow.

1.1 Helmholtz Coil

1.1.1 *Magnetic Field of a Current Loop*

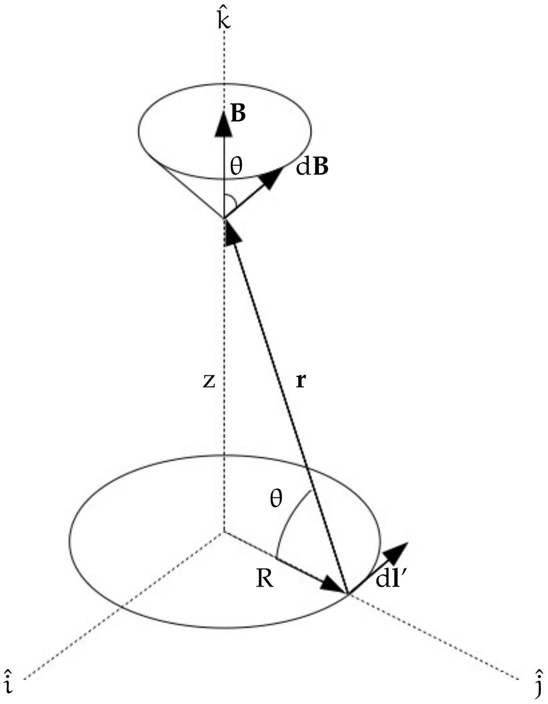


Figure 1: Circular current loop

We want to find the magnetic field at a distance z above the center of circular loop of radius R, which carries a steady current I (see Figure 1). We choose our axes so as to leave the wire loop parallel to the xy plane. Recall that in polar coordinates the unit vectors $\hat{\rho}, \hat{\phi}, \hat{z}$ follow the properties

$$\hat{\rho} \times \hat{\phi} = \hat{z} \longrightarrow \hat{\phi} \times \hat{\rho} = -\hat{z}$$
$$\hat{\phi} \times \hat{z} = \hat{\rho} \longrightarrow \hat{z} \times \hat{\phi} = -\hat{\rho}$$
$$\hat{z} \times \hat{\rho} = \hat{\phi} \longrightarrow \hat{\rho} \times \hat{z} = -\hat{\phi}$$

Clearly, the element $\mathbf{dl'}$ is given by $\mathbf{dl'} = dl\hat{\phi}$. Now, suppose that $z = 0$. This implies $\hat{\mathbf{r}}$ always points inward, or $\hat{\mathbf{r}} = -\frac{R\hat{\rho}}{\sqrt{R^2}} = -\hat{\rho}$. A unit vector that points inward *and* in the z direction is then given by $\mathbf{r} = -R\hat{\rho} + z\hat{z}$. Normalizing this vector we get $\hat{\mathbf{r}} = \frac{-R\hat{\rho}+z\hat{z}}{\sqrt{R^2+z^2}}$. Then

$$\begin{aligned}
d\mathbf{l'} \times \hat{\mathbf{r}} &= \frac{dl}{\sqrt{R^2+z^2}}\left[\phi \times (-R\hat{\rho} + z\hat{z})\right] \\
&= \frac{dl}{\sqrt{R^2+z^2}}\left[R\hat{z} + z\hat{\rho}\right]
\end{aligned}$$

and given that $|r|^2 = R^2 + z^2$ we find that

$$\begin{aligned}
\frac{d\mathbf{l'} \times \hat{\mathbf{r}}}{|r|^2} &= \frac{dl}{(R^2+z^2)^{3/2}}\left[R\hat{z} + z\hat{\rho}\right] \\
\int \frac{d\mathbf{l'} \times \hat{\mathbf{r}}}{|r|^2} &= \int \frac{dlR}{(R^2+z^2)^{3/2}}\hat{z} + \int \frac{dlz}{(R^2+z^2)^{3/2}}\hat{\rho}
\end{aligned}$$

and given that $dl = Rd\phi$

$$\int \frac{d\mathbf{l'} \times \hat{\mathbf{r}}}{|r|^2} = \frac{R^2}{(R^2+z^2)^{3/2}}\int \hat{z}d\phi + \frac{Rz}{(R^2+z^2)^{3/2}}\int \hat{\rho}d\phi$$

Observe that, given that $\hat{\rho} = \cos\phi\hat{\imath} + \sin\phi\hat{\jmath}$, around a close loop:

$$\begin{aligned}
\int_0^{2\pi} \hat{\rho}d\phi &= \int_0^{2\pi} \cos\phi\hat{\imath}d\phi + \int_0^{2\pi} \sin\phi\hat{\jmath}d\phi \\
&= \hat{\imath}\sin\phi - \hat{\jmath}\cos\phi \quad \Big|_0^{2\pi} \\
&= 0
\end{aligned}$$

therefore

$$\begin{aligned}
\int \frac{d\mathbf{l'} \times \hat{\mathbf{r}}}{|r|^2} &= \frac{R^2}{(R^2+z^2)^{3/2}}\int \hat{z}d\phi \\
&= \frac{2\pi R^2}{(R^2+z^2)^{3/2}}
\end{aligned}$$

Consequently

$$\boxed{\mathbf{B(r)} = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}} \tag{2}$$

Of course, we could have applied to symmetry and say that as we integrate $d\mathbf{l}'$ around a loop (refer to Figure 1), $d\mathbf{B}$ sweeps out a cone. The horizontal components cancel, and the vertical components combined to give

$$B(z) = \frac{\mu_0 I}{4\pi}\int \frac{dl'}{r^2}\cos\theta$$

Now, $\cos\theta$ and r^2 are constants, and $\int dl'$ is simply the circumference, $2\pi R$, so

$$B(z) = \frac{\mu_0 I}{4\pi}\left(\frac{\cos\theta}{r^2}\right)2\pi R = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}$$

It's useful to gain experience working out the direction of the field fields and verifying results that arise due to symmetry. After all, not all cases will always be perfectly symmetrical.

1.1.2 *Helmholtz Coil*

The magnetic field on the axis of a circular loop (Equation 2) is far from uniform (it falls off sharply with increasing z). You can produce a more nearly uniform field by using two such loops a distance d apart. We offset the center of each loop by d/2 and set $z = 0$ to be halfway between them. Then, the net field on the axis is, by the principle of superposition:

$$B(z) = \frac{\mu_0 I R^2}{2}\left[\frac{1}{(R^2+(z-d/2)^2)^{3/2}} + \frac{1}{(R^2+(z+d/2)^2)^{3/2}}\right]$$

The derivative of B is

$$\frac{\partial B(z)}{\partial z} = -\frac{3}{2}\frac{\mu_0 I R^2}{2}\left[\frac{2z-d}{(R^2+(z-d/2)^2)^{5/2}} + \frac{2z+d}{(R^2+(z+d/2)^2)^{5/2}}\right]$$

Observe that the derivative is always zero at $z = 0$. Now the second derivative is

$$\begin{aligned}\frac{\partial^2 B(z)}{\partial z^2} &= \frac{15}{4}\frac{\mu_0 I R^2}{2}\left[\frac{(2z-d)^2}{(R^2+(z-d/2)^2)^{7/2}} + \frac{(2z+d)^2}{(R^2+(z+d/2)^2)^{7/2}}\right] \\ &\quad - \frac{3}{2}\frac{\mu_0 I R^2}{2}\left[\frac{2}{(R^2+(z-d/2)^2)^{5/2}} + \frac{2}{(R^2+(z+d/2)^2)^{5/2}}\right]\end{aligned}$$

at $z = 0$

$$\begin{aligned}\left.\frac{\partial^2 B(z)}{\partial z^2}\right|_{z=0} &= \frac{15}{4}\frac{\mu_0 I R^2}{2}\left[2\frac{d^2}{(R^2+(d/2)^2)^{7/2}}\right] - \frac{3}{2}\frac{\mu_0 I R^2}{2}\left[2\frac{2}{(R^2+(d/2)^2)^{5/2}}\right] \\ &= \frac{\mu_0 I R^2}{2}\left[\frac{15}{2}\frac{d^2}{(R^2+(d/2)^2)^{7/2}} - \frac{6}{(R^2+(d/2)^2)^{5/2}}\right]\end{aligned}$$

imposing the condition $\left.\frac{\partial B(z)}{\partial z}\right|_{z=0} = 0$ we find that

$$\begin{aligned}0 &= \left[\frac{15}{2}\frac{d^2}{(R^2+(d/2)^2)^{7/2}} - \frac{6}{(R^2+(d/2)^2)^{5/2}}\right] \\ &= \left[\frac{15}{2}d^2 - 6R^2 - 6(d/2)^2\right] \\ &= \left[(\frac{30}{4} - \frac{6}{4})d^2 - \frac{24}{4}R^2\right] \\ &= d^2 - R^2\end{aligned}$$

Thus, the second derivative vanishes if $d = R$, in which case the field at the center is

$$\boxed{B(z=0) = \left(\frac{4}{5}\right)^{3/2}\frac{\mu_0 I}{R} \quad \text{if } d = R} \tag{3}$$

For practical purposes the use of several turns in each coil is used. If each coil has N turns the field is simply given by $B_T(z=0) = NB(z=0)$ which can be conveniently expressed as $B = 9.0 \times 10^{-7}\frac{NI}{R}$.

1.1.3 *Charge to Mass*

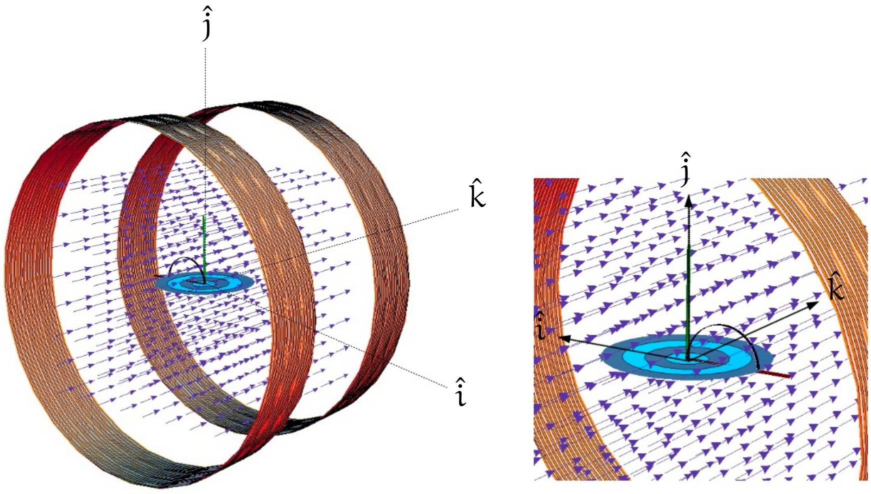


Figure 2: Electron inside the magnetic field of a Helmholtz coil

Consider a pair of Helmholtz coils both with a radius R and separated by a distance $d = R$ such that the magnetic field at the center ($z = 0$) is given by Equation 3 and it's on the $\hat{k}$ direction. In the center of the coils there is an electron tube from which electrons emerged (see Figure 2) with velocity $\vec{v} = v\hat{\jmath}$. By the Lorentz force law, the electron at $z = 0$ experiences a force given by

$$\begin{aligned}\vec{F}_{z=0} &= -e\vec{v} \times \vec{B} \\ &= -evB\hat{\jmath} \times \hat{k} \\ &= -evB\hat{\imath}\end{aligned}$$

The archetypical motion of a charged particle in a magnetic field is circular, with the magnetic force providing the centripetal acceleration as the electron changes velocity (the magnitude of v is unchanged, as B only projects some of the speed onto the $-\hat{\imath}$ component). Newton's second law gives $evB = mv^2/r$, or

$$v = \frac{erB}{m}$$

To find an expression for v in terms of quantities that can be measured, we note that the kinetic energy imparted to an electron within the tube is given by eV where V si the potential difference through which the electrons have been accelerated and is the voltage between the cathode and anode. Then we get

$$eV = \frac{1}{2}mv^2$$

therefore

$$\frac{e}{m} = \frac{2V}{r^2B^2}$$

Combining this with Equation 3 we get for an N number of turns

$$\boxed{\frac{e}{m} = \frac{V}{I^2r^2}\left[\left(\frac{5}{4}\right)^3 \frac{2R^2}{N^2\mu_0^2}\right]} \tag{4}$$

Note that r is the radius of the electron's orbit and R is the radius of the coils. The constants R, N, and μ_0 can be measured while different values of V, I, and r must be obtained as they are dependent on each other.

1.2 Experiment

To start the experiment, consult with the instructor to see if the cables are connected correctly. Then, activate the cathode check if you can see a blue line of electrons straight upward in the glass chamber. To get the beam focused, you should raise the cathode's accelerating voltage until the electrons were only negligibly scattered. Then, switch on the power for the Helmholtz coils. They produce identical magnetic fields perpendicular to their path that complement each other greatest at the center between them, where we'd set up our electron beam. Depending on the current's orientation, the electrons are either deflected in a soft arc toward or away from the viewer. A cardboard box with a viewing slit should be utilized to diminish outside light and make the electron beam more visible.

There are marker circles on the fluorescent screen. Make the electron beam hit these circles by changing acceleration voltage and current through the coils. This way you can measure the diameter of the electron beam path. Make several measurements.

1.2.1 *Tasks*

- Measure the magnetic field of the Helmholtz coils using the hand-held apparatus provided. Hold the magnetometer in different orientations to see in which direction the magnetic field lines are aligned. How does the magnetometer utilizes the Hall effect? Compare the measured magnetic field against the calculated field.
- Determine the charge to mass ratio.

1.2.2 *Preparation key words*

Calculation of e/m, magnetic field induced by a current, fields by coils, Biot-Savart law, Maxwell equations, Hall effect.

REFERENCES

[1] Llewellyn, Ralph and Tipler, Paul *Modern Physics* 4th ed. New York: W.H. Freeman and company, 2003

[2] Shankar, Ramaruti *Principles of Quantum Mechanics* 2nd ed. New York: Springer, 1994.

[3] Serway, Raymond, John Jewett, Jr *Physics For Scientists and Engineers* Vol. 2. California: Brooks and Cole, 2004.

2 THE FRANCK-HERTZ EXPERIMENT

The Franck-Hertz experiment was done in 1914. It was the first direct measurement of quantization of energy levels in an atom1. It showed a set of steps in the energy of a system that could only be explained at the time with Bohr's model of the atom which was proposed, unknowingly to Franck and Hertz, a year earlier. Fig. 1 shows the experimental set up basically consisting of a vacuum tube filled with mercury vapor, a heated cathode, an anode, and a wire mesh placed in between the two nodes. A potential difference is applied between the plates and the mesh grid. The first potential difference is used to accelerate the electrons through the vapor. The second only allows electrons with some kinetic energy to collide with the collector.

The Cathode produces electrons that accelerate towards the mesh grid and then collide with the collecting plate producing a current. The voltage could be adjusted to create a larger acceleration and therefore was expected to produce a larger current.

In the experiment special features were observed: if the electrons energy exceeds a value of 4.9 eV the current dropped suddenly. This effect was repeated at integer multiples of 4.9 eV. This is explained by the quantum nature of the mercury vapor. At these energies the electrons are no longer colliding with the atoms elastically. This energy of 4.9 eV is needed to excite a mercury atom. The electron is then released after the excitation with less kinetic energy due to a photon being emitted when the atom returns to the non-excited state. The integer repetition is a result of the released electron having another 4.9 eV to excite a mercury atom again.

2.1 Experiment

Our experiment is very similar to the one described above. The Franck-Hertz tube is inside an oven that has to be maintained within a certain temperature range. The oven assures that mercury stays in vapor form, which does not occur at lower temperatures. An oscilloscope and the accelerating voltage supply allow for the data acquisition.

The first run of the experiment should be done with a temperature setting of 190oC. The temperature of the unit should be allowed to rise slowly and then settle within this range. A periodic signal with linearly increasing voltage is supplied. The current is to be measured as a function of voltage. First, make yourself familiar with the operation of the oscilloscope. Show the input voltage as a function of time, then show the output current as function of time.

Use the x-y mode to display the current as a function of voltage. Take measurements of current and voltage at different points of the curve to accurately describe the measured curve.

The potential can be raised till the current drop of each integer voltage is reached. This procedure can be used to get an accurate reading for the current drops. Change the oven temperature to 180oC and take another measurement. How does the oven temperature change the measured curve.

What do you expect for even lower temperatures, and for higher temperatures? Why do we not see any light being emitted from the experiment?

2.1.1 *Preparation key words*

Bohr's model of the atom, energy levels of electrons in an atom, occupied and unoccupied states, kinetic and potential energy, vapor pressure, thermal activation

REFERENCES

[1] Llewellyn, Ralph and Tipler, Paul *Modern Physics* 4th ed. New York: W.H. Freeman and company, 2003

[2] Valerie Gordeski *The Frank-Hertz Experiment Presentation* Massachusetts Institute of Technology

[3] Department of Physics and Astronomy *Host of Hyperphysics, Georgia St. University* http://hyperphysics.phy-astr.gsu.edu/hbase/FrHz.html

3 OPTICAL DIFFRACTION

3.1 Definitions

Definition 3.1. The **principle of superposition** states that when two or more waves overlap, the resultant displacement at any point and at any instant is found by adding the instantaneous displacements that would be produced at the point by the individual waves if each were present alone.

Definition 3.2. Two monochromatic (single frequency) sources of the same frequency and with a constant phase relation (not necessarily in phase) are said to be **coherent**.

Definition 3.3. Let source S_1 be a distance r_1 from point P and let source S_2 be a distance r_2 from point P.

- **Constructive interference** $\longrightarrow$ $r_2 - r_1 = m\lambda$
- **Constructive interference** $\longrightarrow$ $r_2 - r_1 = (m + 1/2)\lambda$

where $m = 0, \pm 1, \pm 2, ...$

Definition 3.4. According to **Huygen's Principle**, we can consider every point of a wave as a source of secondary wavelets. These spread out in all directions with a speed equal to the speed of propagation of the wave. The position of the front at any later time is the "envelope" of the secondary wavelets at that time.

Definition 3.5. Fresnel diffraction, or *near-field* diffraction, refers to when both the point source and screen are relatively close to the obstacle forming the diffraction pattern.

Definition 3.6. Fraunhofer diffraction, or *Franhofer limit* diffraction, refers to situations in which the source, obstacle, and screen are far enough apart that we can consider all lines from the source to the obstacle to be parallel, and can likewise consider all lines from the obstacle to a given point on the screen to be parallel.

Definition 3.7. The **time-average** expectation value of an electromagnetic wave with period T is given by

$$< E^2 >= \frac{1}{T}\int_0^T |E|^2 dt = \frac{1}{T}\int_0^T E^* E dt \tag{5}$$

where E^* is the complex conjugate of E.

Definition 3.8. The **intensity** of an electromagnetic wave is given by

$$I = K < E^2 > \tag{6}$$

where K is a constant and $< E^2 >$ is the time-averaged expectation value of the electromagnetic wave.

3.2 Complex Sum

Consider the sum

$$\sum_{n=0}^{N-1} e^{in\delta} = 1 + e^{i\delta} + e^{2i\delta} + \cdots + e^{i(N-1)\delta}$$

multiplying by $e^{i\delta}$

$$e^{i\delta}\sum_{n=0}^{N-1} e^{in\delta} = e^{i\delta} + e^{2i\delta} + e^{3i\delta} + \cdots + e^{iN\delta}$$

then

$$\sum_{n=0}^{N-1} e^{in\delta} - e^{i\delta}\sum_{n=0}^{N-1} e^{in\delta} = \sum_{n=0}^{N-1} e^{in\delta}\left(1 - e^{i\delta}\right) = 1 - e^{iN\delta}$$

consequently

$$\sum_{n=0}^{N-1} e^{in\delta} = \frac{1 - e^{iN\delta}}{1 - e^{i\delta}}$$

which is logically equivalent to

$$\begin{aligned}\sum_{n=0}^{N-1} e^{in\delta} &= \frac{1 - e^{iN\delta}}{1 - e^{i\delta}} \cdot \frac{1 - e^{-i\delta}}{1 - e^{-i\delta}}\\ &= \frac{1 - e^{-i\delta} - e^{iN\delta} + e^{iN\delta}\cdot e^{-i\delta}}{1 - e^{-i\delta} - e^{+i\delta} + 1}\end{aligned}$$

Note that $1 = e^{i\frac{N}{2}\delta}\cdot e^{-i\frac{N}{2}\delta} = e^{i\frac{1}{2}\delta}\cdot e^{-i\frac{1}{2}\delta}$, $e^{iN\delta} = e^{i\frac{N}{2}\delta}\cdot e^{i\frac{N}{2}\delta}$, and $e^{-i\delta} = e^{-i\frac{1}{2}\delta}\cdot e^{-i\frac{1}{2}\delta}$, therefore we can rewrite the sum as

$$\begin{aligned}\sum_{n=0}^{N-1} e^{in\delta} &= \frac{1}{2 - 2\frac{e^{i\delta}+e^{-i\delta}}{2}}\Big[\left(e^{i\frac{N}{2}\delta}\cdot e^{-i\frac{N}{2}\delta}\cdot e^{i\frac{1}{2}\delta}\cdot e^{-i\frac{1}{2}\delta}\right) -\\ &\qquad \left(e^{i\frac{N}{2}\delta}\cdot e^{-i\frac{N}{2}\delta}\cdot e^{-i\frac{1}{2}\delta}\cdot e^{-i\frac{1}{2}\delta}\right) -\\ &\qquad \left(e^{i\frac{N}{2}\delta}\cdot e^{i\frac{N}{2}\delta}\cdot e^{i\frac{1}{2}\delta}\cdot e^{-i\frac{1}{2}\delta}\right) +\\ &\qquad \left(e^{i\frac{N}{2}\delta}\cdot e^{i\frac{N}{2}\delta}\cdot e^{-i\frac{1}{2}\delta}\cdot e^{-i\frac{1}{2}\delta}\right)\Big]\\ &= \frac{e^{i\frac{N}{2}\delta}e^{-i\frac{1}{2}\delta}}{2 - 2\cos\delta}\Big[\left(e^{-i\frac{N}{2}\delta}\cdot e^{i\frac{1}{2}\delta}\right) - \left(e^{-i\frac{N}{2}\delta}\cdot e^{-i\frac{1}{2}\delta}\right) -\\ &\qquad \left(e^{i\frac{N}{2}\delta}\cdot e^{i\frac{1}{2}\delta}\right) + \left(e^{i\frac{N}{2}\delta}\cdot e^{-i\frac{1}{2}\delta}\right)\Big]\\ &= \frac{e^{i\frac{\delta}{2}(N-1)}}{2(1-\cos\delta)}\left[-\left(e^{i\frac{N}{2}\delta} - e^{-i\frac{N}{2}\delta}\right)\left(e^{i\frac{1}{2}\delta} - e^{-i\frac{1}{2}\delta}\right)\right]\end{aligned}$$

$$= -\frac{e^{i\frac{\delta}{2}(N-1)}}{\sin^2\frac{\delta}{2}}\sin(\frac{N\delta}{2})\sin(\frac{\delta}{2})$$

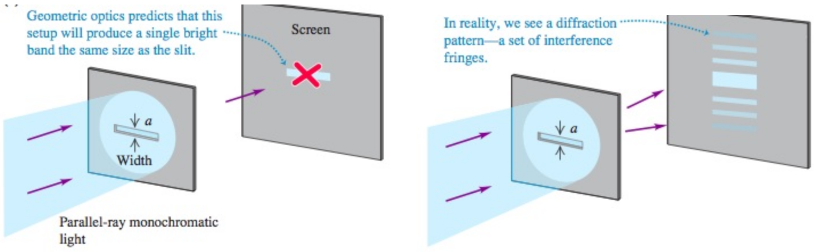
Therefore

$$\boxed{\sum_{n=0}^{N-1} e^{in\delta} = -e^{i\frac{\delta}{2}(N-1)}\frac{\sin\frac{N\delta}{2}}{\sin\frac{\delta}{2}}} \tag{7}$$

3.3 Diffraction

Diffraction is a general term for interference effects related to edges or apertures. Diffraction is more familiar in waves with longer wavelengths than those of light. For example, diffraction is what causes sound to bend around corners or spread as it passes through a doorway. Water waves spread as they pass between rocks near a rugged coast because of diffraction. Two different regimes for diffraction are usually identified: Fresnel and Fraunhofer.

The phenomenon of diffraction can be understood using Huygens's principle (see Definition 3.4). According to Huygens's principle, light waves incident on slits will spread out and exhibit an interference pattern in the region beyond . The pattern is called a diffraction pattern (see Figure 3b). On the other hand, if no bending occurs and the light wave continue to travel in straight lines, then no diffraction pattern would be observed (see Figure 3a).



(a) Predicted outcome **(b)** What really happens

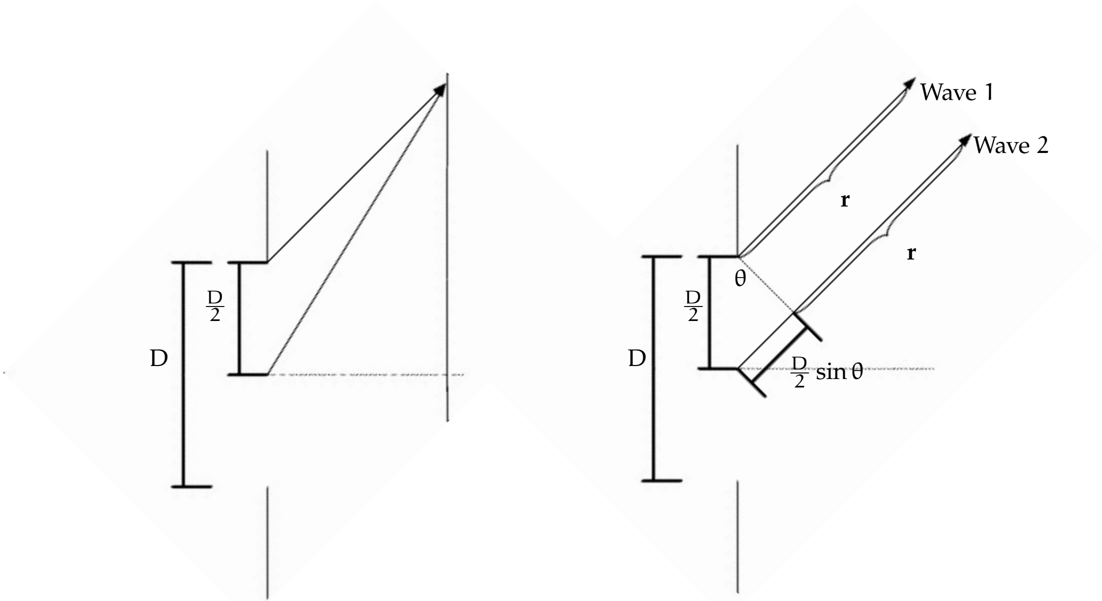
Figure 3: Single slit diffraction[?].

3.3.1 *Single Slit Diffraction: Dark Fringes*

Consider a plane wave incident of a single slit as shown on Figure 4. In diffraction of Fraunhofer type (see Definition **??**), all rays passing through the slit are approximately parallel. In addition, each portion of the slit will act as a source of light waves according to Huygens's principle. For simplicity we divide the slit into two halves.

The difference in path length Δr to a point P far away from the source (Fraunhofer limit) is $D/2 \sin\theta$, where D is the slit width and θ is the angle between the perpendicular to the slit and a line from the center of the slit to P. Suppose now that this path difference happens to be equal to $\lambda/2$; then light from these two strips arrives at point P with half-cycle phase difference, and cancellation occurs.

Similarly, we could have chosen to work with the light from two strips immediately *below* the two points in the figure that also arrives at P. These two half have the same path *difference* therefore they also arrive at point P



Set up Fraunhofer limit

Figure 4: Single slit diffraction

a half-cycle out of phase. In fact, the light from every strip in the top half of the slit cancels out the light rom a corresponding strip in the bottom half. This implies that there is a dark fringe whenever

$$\frac{D}{2}\sin\theta = \pm\frac{\lambda}{2}$$

or

$$\sin\theta = \pm\frac{\lambda}{D}$$

But Huygen's principle says that we can treat *every* element of the area as sources, therefore we could divide the screen into quarters, sixths, and so on, and use the above argument to show that the condition for a dark fringe is

$$\boxed{\sin\theta = \frac{m\lambda}{D} \quad (m = \pm1, \pm2, ...) \quad \textit{Dark Fringe}} \tag{8}$$

3.3.2 *Single Slit Diffraction: Intensity*

Consider a single slit divided up into N very thin strips of width Δy (see Figure 5). Note that $D = N\Delta y$ is an approximation which becomes exact in the limit $N \to \infty$, $\Delta y \to 0$ while keeping the product $D = N\Delta y$ constant.

We choose the strip width such that $\Delta y << \lambda$ so that all the light from a given strip is in phase. The strips are of equal size, and if the whole strip is uniformly illuminated, we can take the electric field wave amplitudes from each strip to be equal, i.e.

$$\Delta E_0 = \frac{E_0}{N} \tag{9}$$

where E_0 is the original amplitude and ΔE_0 is the amplitude of each strip wave.

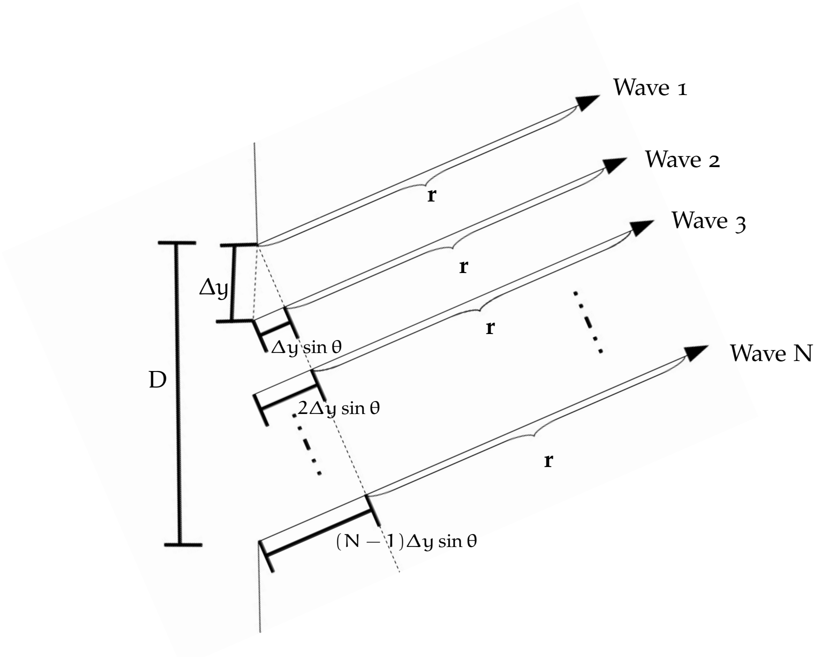


Figure 5: N Slit diffraction

Consider wave 1 and wave 2. At any given point on a screen far away (Franhofer limit), wave 1 travels a distance $r_1 = r$ while wave 2 travels a distance $r_2 = r + \delta r$, where $\delta r = \Delta y \sin\theta$ is the *path* difference. This implies a *phase* difference given by $\delta = k\Delta r = k(r_2 - r_1) = k\delta r = k\Delta y \sin\theta$ where $k = 2\pi/\lambda$ is the wave number. In general, the phase difference of each wave after wave 1 is given by the n multiple of

$$\delta = \frac{2\pi\Delta y \sin\theta}{\lambda} \tag{10}$$

The amplitude of each strip wave is ΔE_0, therefore we can express each individual wave at a point x by

$$\begin{aligned}
E_1(x,t) &= \Delta E_0 e^{i(kx+\omega t)} \\
E_2(x,t) &= \Delta E_0 e^{i(kx+\omega t+\delta)} \\
E_3(x,t) &= \Delta E_0 e^{i(kx+\omega t+2\delta)} \\
&\vdots \\
E_N(x,t) &= \Delta E_0 e^{i(kx+\omega t+(N-1)\delta)}
\end{aligned}$$

The total electric field at a point P is given by the total contribution of each wave, i.e.

$$E(x,t) = \Delta E_0 \sum_{n=0}^{N-1} e^{i(kx+\omega t+n\delta)}$$

by Equation 9 $\Delta E_0 = E_0/N$, therefore

$$E(x,t) = \frac{1}{N} E_0 e^{i(kx+\omega t)} \sum_{n=0}^{N-1} e^{in\delta}$$

The sum is given by Equation 7. Then

$$E(x,t) = -\frac{1}{N} E_0 e^{i(kx+\omega t)} e^{i\frac{\delta}{2}(N-1)} \frac{\sin\frac{N\delta}{2}}{\sin\frac{\delta}{2}}$$

For small angles θ, $\sin\theta \approx \theta$, therefore by Equation 10 δ is also small and we may approximate $\sin\frac{\delta}{2} \approx \frac{\delta}{2}$ (note that N in general may be very large and it would be wrong to assume that the product $\frac{N\delta}{2}$ is small). Then

$$E(x,t) = -\frac{1}{N} E_0 e^{i(kx+\omega t)} e^{i\frac{\delta}{2}(N-1)} \frac{\sin\frac{N\delta}{2}}{\frac{\delta}{2}}$$

$$E(x,t) = -E_0 e^{i(kx+\omega t)} e^{i\frac{\delta}{2}(N-1)} \frac{\sin\frac{N\delta}{2}}{\frac{N\delta}{2}}$$

By Equation 10, $N\delta/2 = \pi N\Delta y \sin\theta/\lambda$. But $N\Delta y = D$, therefore $N\delta/2 = \pi D \sin\theta/\lambda$. Thus

$$E(x,t) = -E_0 e^{i\left(kx+\omega t+\frac{\delta}{2}(N-1)\right)} \frac{\sin\left[\frac{\pi D}{\lambda}\sin\theta\right]}{\frac{\pi D}{\lambda}\sin\theta}$$

By Definition 3.7, the time-average of E is

$$\begin{aligned} < E_\theta^2 > &= \frac{1}{T}\int_0^T \left|E(x,t)\right|^2 dt \\ &= \frac{1}{T}E_0^2 \int_0^T \left(\frac{\sin\left[\frac{\pi D}{\lambda}\sin\theta\right]}{\frac{\pi D}{\lambda}\sin\theta}\right)^2 dt \end{aligned}$$

since for any complex number z, $|e^{iz}| = e^{iz}\cdot e^{-iz} = 1$. Then

$$< E_\theta^2 > = \left(E_0 \frac{\sin\left[\frac{\pi D}{\lambda}\sin\theta\right]}{\frac{\pi D}{\lambda}\sin\theta}\right)^2$$

By Definition 3.8, $I_0 = K < E_0^2 >= KE_0^2$. Then, the intensity as a function of θ is given by

$$\boxed{I(\theta) = I_0\left(\frac{\sin\left[\frac{\pi D}{\lambda}\sin\theta\right]}{\frac{\pi D}{\lambda}\sin\theta}\right)^2} \qquad (11)$$

3.3.3 *Single Slit Diffraction: Discussion*

The intensity drops to 0 when $\sin\left[\frac{\pi D}{\lambda}\sin\theta\right] = 0$. This can occur only when $\frac{\pi D}{\lambda}\sin\theta = 0$ or $\frac{\pi D}{\lambda}\sin\theta = m\pi$ $(m = \pm1, \pm2, ...)$.

- If $\frac{\pi D}{\lambda}\sin\theta = 0$ we also have a 0 in the denominator. Then, by $l'hopital'srule$,

$$\lim_{\frac{\pi D}{\lambda}\sin\theta\to 0} \frac{\sin\left[\frac{\pi D}{\lambda}\sin\theta\right]}{\frac{\pi D}{\lambda}\sin\theta} = 1$$

and therefore $I(\theta = 0) = I_0$ is a max

.

- $\frac{\pi D}{\lambda}\sin\theta = m\pi$ implies that there is a dark fringe whenever

$$\sin\theta = m\frac{\lambda}{D} \qquad m = \pm 1, \pm 2, \ldots$$

in agreement with equation 8

The angle θ is small only if the wavelength is small compared to the slit width. The plot of Equation 11 is shown in Figure 6 for the typical case where $\lambda = 500\,\text{nm}$ and $D = 10^{-2}\,\text{cm}$, or $\lambda/D = 1/200$ (which implies θ is small).

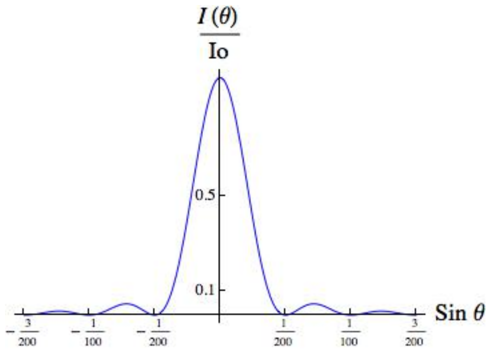


Figure 6: Diffraction intensity plot

As expected, for $\theta = 0 \to \sin\theta = 0$ the intensity is I_0.

3.4 Interference

In order to form an interference pattern, the incident light must be *coherent* and *monochromatic*. For example, light emitted from an incandescent lightbulb is incoherent because the light consists o waves of different wavelengths and they do not maintain a constant phase relationship. Thus, no interference pattern is observed.

In 1801 Thomas Young carried out an experiment in which the wave nature of light was demonstrated. The schematic diagram of the double-slit experiment is shown in Figure 7. A monochromatic light source is incident on the first screen which contains a slit S_0. The emerging light then arrives at the second screen which has two parallel slits S_1 and S_2. which serve as the sources of coherent light. The light waves emerging from the two slits then interfere and form an interference pattern on the viewing screen. The bright bands (fringes) correspond to interference maxima, and the dark band interference minima.

3.4.1 *Double Slit Interference*

Figure 8 shows the double slit interference schematic in the Franhofer limit. For any particular point P on the screen, wave 1 travels $r_1 = r$ and wave 2 travels $r_2 = r + \delta r$, where $\delta r = d\sin\theta$ is the path difference. The condition for constructive interference requires that $r_2 - r_1 = m\lambda$ $m = 0, \pm 1, \pm 2, \ldots$, therefore

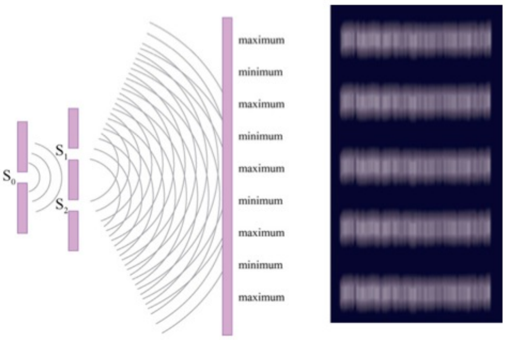
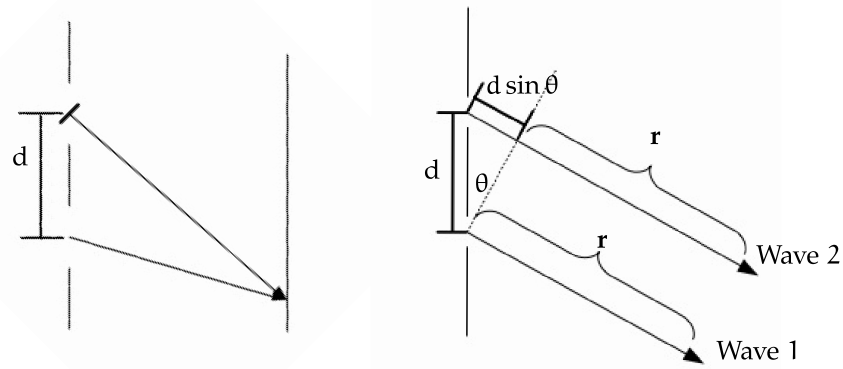


Figure 7: Young's double slit experiment

$$r_2 - r_1 = d \sin\theta$$

or

$$\boxed{d \sin\theta = m\lambda \quad (m = 0, \pm 1, \pm 2, \ldots) \quad \textit{Bright Fringe}} \tag{12}$$



Set up

Fraunhofer limit

Figure 8: Interference set up

Because wave 1 and wave 2 have a *path* difference δr when they arrive at point P, this implies that they have a *phase* difference δ given by $\delta r = k\Delta r = k(r_2 - r_1) = k\delta r = kd \sin\theta$, where $k = 2\pi/\lambda$ is the wave number.

$$\delta = \frac{2\pi d \sin\theta}{\lambda} \tag{13}$$

We assume the strips are of equal size, and if the whole strip is uniformly illuminated, we can take the electric field wave amplitudes E_0 from each strip to be equal. Then,

$$\begin{aligned} E_1(x,t) &= \Delta E_0 e^{i(\omega t + kr)} \\ E_2(x,t) &= \Delta E_0 e^{i(\omega t + kr + \delta)} \end{aligned}$$

The electric field at point P is given by the sum of both fields, i.e.

$$E(x,t) = E_0 e^{i(\omega t + kr)} + E_0 e^{i(\omega t + kr + \delta)}$$

For any complex z, $|e^{iz}| = e^{iz} \cdot e^{-iz} = 1$. Then,

$$\begin{aligned} \left|E(x,t)\right|^2 &= E_0^2\left(1 + 1 + e^{-1\delta} + e^{\delta}\right) \\ &= E_0^2\left(2 + 2\cos\delta\right) \end{aligned}$$

and given the identity $1 + \cos\alpha = 2\cos\frac{\alpha}{2}$

$$\left|E(x,t)\right|^2 = 4E_0^2 \cos^2\frac{\delta}{2}$$

By Definition 3.7, the time-average of E is

$$\begin{aligned} < E_\theta^2 > &= \frac{1}{T}\int_0^T \left|E(x,t)\right|^2 dt \\ &= \frac{4}{T} E_0^2 \int_0^T \cos^2\frac{\delta}{2} dt \end{aligned}$$

thus

$$< E_\theta^2 > = 4E_0^2 \cos^2\frac{\delta}{2}$$

By Definition 3.8, $I_0 = K < E_0^2 >= KE_0^2$. Then, the intensity as a function of θ is given by

$$\boxed{I(\theta) = 4I_0\cos^2\frac{\delta}{2} \quad \textit{Double Slit Intensity}} \tag{14}$$

3.4.2 *N Slit Interference*

Figure 9 shows a generalization to N slits. Each wave may be described by

$$\begin{aligned} E_1(x,t) &= E_0 e^{i(\omega t + kr)} \\ E_2(x,t) &= E_0 e^{i(\omega t + kr + \delta)} \\ E_3(x,t) &= E_0 e^{i(\omega t + kr + 2\delta)} \\ &\vdots \\ E_N(x,t) &= E_0 e^{i(\omega t + kr + (N-1)\delta)} \end{aligned}$$

or equivalently

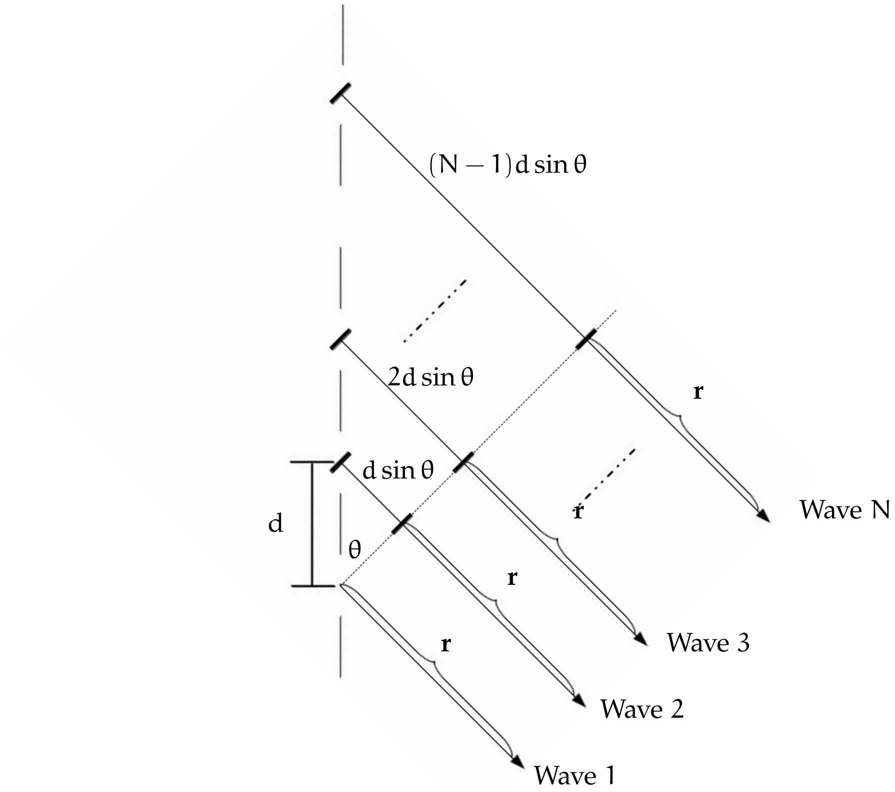


Figure 9: N Slit Interference

$$E_N(x,t) = \sum_{n=0}^{N-1} E_0 e^{i(\omega t + kr + n\delta)}$$

The sum is given by Equation 7. Then

$$E(x,t) = E_0 e^{i(kx+\omega t)} e^{i\frac{\delta}{2}(N-1)} \frac{\sin\frac{N\delta}{2}}{\sin\frac{\delta}{2}}$$

By Definition 3.7, the time-average of E is

$$\begin{aligned} < E_\theta^2 > &= \frac{1}{T}\int_0^T |E(x,t)|^2 dt \\ &= \frac{1}{T} E_0^2 \int_0^T \left(\frac{\sin\left[\frac{\pi dN}{\lambda}\sin\theta\right]}{\frac{\pi dN}{\lambda}\sin\theta} \right)^2 dt \end{aligned}$$

since for any complex number z, $|e^{iz}| = e^{iz} \cdot e^{-iz} = 1$. Then

$$< E_\theta^2 > = \left(E_0 \frac{\sin\left[\frac{\pi dN}{\lambda}\sin\theta\right]}{\frac{\pi dN}{\lambda}\sin\theta} \right)^2$$

By Definition 3.8, $I_0 = K < E_0^2 >= KE_0^2$. Then, the intensity as a function of θ is given by

$$\boxed{I(\theta) = I_0\left(\frac{\sin\left[\frac{\pi\, dN}{\lambda}\sin\theta\right]}{\frac{\pi\, dN}{\lambda}\sin\theta}\right)^2 \quad \textit{N Slit Intensity}} \tag{15}$$

It can be shown that Equation 15 reduces to Equation 14 for $N = 2$.

3.4.3 *Discussion*

For two sources with electric fields E_1 and E_2, the total instantaneous electric field E at the point P on the screen is equal to the vector sum of the two sources: $E = E_1 + E_2$. On the other hand, the Poynting flux S is proportional to the square of the total field:

$$S \propto |E|^2 = |E_1 + E_2|^2 = |E_1|^2 + |E_2|^2 + 2E_1 \cdot E_2$$

Taking the time average of S, the intensity I on the light at P may be obtained as

$$I = < S > \propto < E_1^2 > + < E_2^2 > +2 < E_1 \cdot E_2 >$$

The cross term $2 < E_1 \cdot E_2 >$ represents the correlation between the two light waves. For *incoherent* light sources, since there is no definite phase relation between E_1 and E_2, the cross term vanishes, and the intensity due to the incoherent source is simply the sum of the two individual intensities:

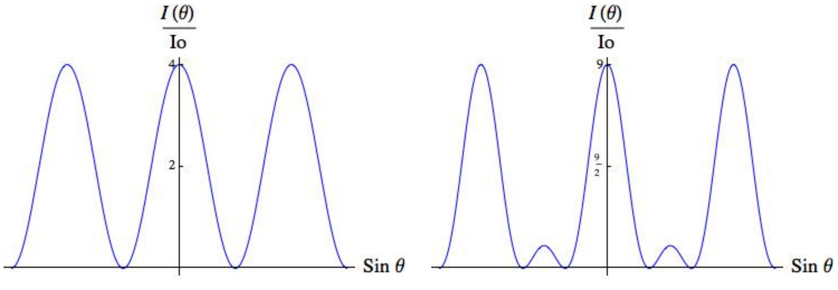
$$I_{incoherent} = I_1 + I_2$$

For coherent sources, the cross term is non-zero. In fact, as it was shown, for *constructive* interference in a two source system, the resulting intensity is

$$I_{coherent} = 2^2 I_0$$

In fact, it can be shown that, in general, for N coherent sources

$$I_{coh} = N^2 I_0$$



(a) $N = 2$ **(b)** $N = 3$

Figure 10: Interference intensity.

3.5 Diffraction and Interference

Suppose that we have two slits, each having a width D, and separated by a distance d. As discussed, both slits will behave as lights sources and we refer to their interaction as interference. Also, each individual slit causes the light to bend and we refer to such effect as diffraction. The intensity for the double slit interference is given by Equation 14 and the intensity for a single slit diffraction is given by 11. The resulting interference pattern for the double slit will also include a diffraction pattern due to the individual slit. The combined effect (for small angles θ) is then

$$I(\theta) = 4I_0 \cos^2 \frac{\pi d \sin\theta}{\lambda} I_0 \left(\frac{\sin\left[\frac{\pi D}{\lambda} \sin\theta\right]}{\frac{\pi D}{\lambda} \sin\theta} \right)^2 \tag{16}$$

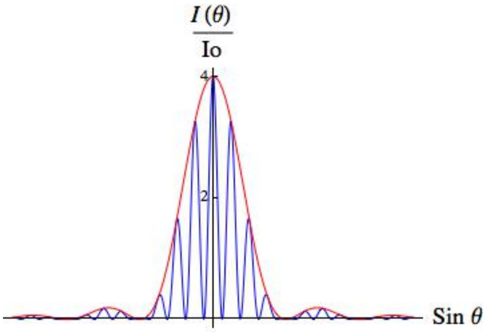
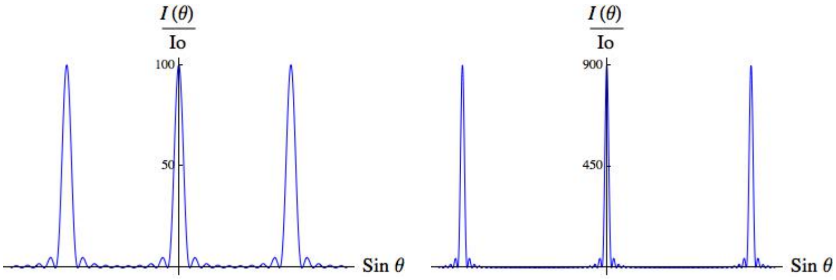


Figure 11: Double slit interference and diffraction intensity.

Figure 11 shows the intensity plot of for a double slit with with slit width of $D = 10^{-2}$ cm and a separation between the widths of $d = 4D$ with a source wavelength of $\lambda = 500$ nm. The first and the second terms in the above equation are referred to as the "interference factor" and the "diffraction factor," respectively. While the former yields the interference substructure, the latter acts as an envelope which sets limits on the number of the interference peaks (see Figure 11).



(a) $N = 10$ (b) $N = 30$

Figure 12: Interference and diffraction intensity.

3.5.1 *Diffraction Gratings*

A diffraction grating consists of a large number N of slits each of width D and separated from the next by a distance d. In general, for N slits:

$$\boxed{I(\theta) = I_0 N^2 \left(\frac{\sin\left[\frac{\pi dN}{\lambda}\sin\theta\right]}{\sin\left[\frac{\pi dN}{\lambda}\sin\theta\right]}\right)^2 \left(\frac{\sin\left[\frac{\pi D}{\lambda}\sin\theta\right]}{\sin\left[\frac{\pi D}{\lambda}\sin\theta\right]}\right)^2} \tag{17}$$

Figure 12 shows the intensity plot for a grating with $N = 10$ and $N = 30$ slits with with slit width of $D = 10^{-2}\,\mathrm{cm}$ and a separation between the widths of $d = 4D$ with a source wavelength of $\lambda = 500\,\mathrm{nm}$. The location of the maxima does not depend on the number of slits, N. However, the maxima become sharper and more intense as N is increased. The width of the maxima can be shown to be inversely proportional to N. Notice in Figure 12 that the principal maxima become sharper and narrower as N increases.

Diffraction gratings are usually label as g = *slits/millimeter*; the number of slits N in a particular grating is then given by N = dg; the value d is then the *reciprocal* of the number of slits per unit length.

Example 3.1. You have been given a grating labeled as $1000\,\mathrm{lines/mm}$. This is equivalent to $1000\,\mathrm{slits/mm}$, or

$$d = \frac{\mathrm{mm}}{1000} = \frac{10^{-3}}{10^3}\,\mathrm{m} = 10^{-6} \quad = 1\,\mu\mathrm{m}$$

which is a typical value for d. Using a red laser of wavelength $\lambda = 650\,\mathrm{nm}$ we expect to find the first order maxima at

$$\begin{aligned} d\sin\theta &= \pm m\lambda = \pm\lambda \\ \theta &= \pm \mathrm{sin}^{-}(\frac{\lambda}{d}) \\ &= \pm \mathrm{sin}^{-}(\frac{650\cdot 10^{-9}}{10^{-6}}) \\ &= \pm \mathrm{sin}^{-}(.650) \\ &= \pm 40.5 \text{ degrees} \end{aligned}$$

Up until now we have assume that the plane waves incident on the slits are parallel to the slits. This is not generally true. Suppose that we have a source S_0 that makes an angle α with the normal of the diffraction grating (see Figure 13). Also, a detector D is placed so that it makes an angle θ with the normal of the diffraction grating.

The bottom wave travels a distance $r + \delta r_1$ and the top travels a distance $r + \delta r_2$, where $\delta r_1 = d\sin\theta$ and $\delta r_2 = d\sin\alpha$. The path difference Δr is then $d(\sin\alpha - \sin\theta)$. Then, the constructive interference requires

$$\boxed{d(\sin\alpha - \sin\theta) = m\lambda \quad m = 0, \pm1, \pm2, \ldots} \tag{18}$$

3.6 Experiment

3.6.1 *Tasks*

- Use the laser to set up an experiment to measure the diffraction of the light on a periodic lattice.

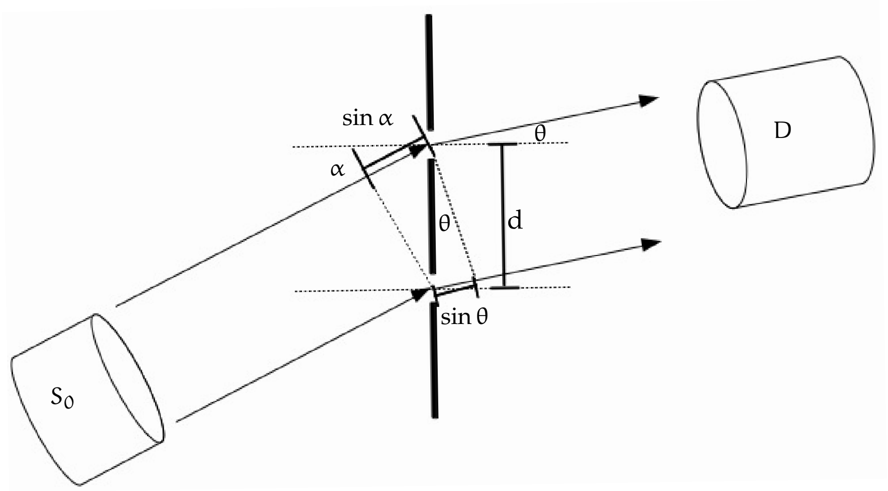


Figure 13: Diffraction grating

- Use lenses to spread the laser beam on an optical bench and produce a parallel beam on the spectrometer.
- Measure the scattering angles of diffraction maxima for the incident beam perpendicular to the optical grating.
- Repeat this measurement for a tilted grating.
- Calculate the lattice spacing if the wavelength is known.

3.6.2 *Preparation key words*

Lasers, characteristic emission lines from gases, absorption, diffraction, interference, coherence, diffraction grating, single slit experiment, Fraunhofer approximation, Huygens principle, explanation of diffraction maxima and minima in the far-field.

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4 AC CIRCUITS

4.1 Review of Classical Circuits

In this section we consider circuits which have a constant (time-independent) source ϵ.

4.1.1 *RC Circuit*

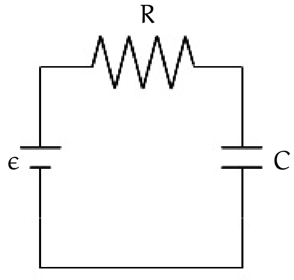


Figure 14: RC Circuit schematic diagram

The underlying principle behind the operation of an RC circuit is given by

$$\oint \vec{E} \cdot \vec{ds} = 0$$

Then

$$\begin{aligned} V_R + V_C =& V_{in} \\ Ri + \frac{q}{C} =& \epsilon \\ \frac{dq}{dt} =& \frac{1}{R}(\epsilon - \frac{q}{C}) \\ =& -\frac{1}{RC}(q - \epsilon C) \end{aligned}$$

rearranging and integrating

$$\begin{aligned} \int_{q_0}^{q} \frac{dq'}{q' - \epsilon C} =& -\frac{1}{RC}\int_0^t dt \\ \ln(\frac{q - \epsilon C}{q_0 - \epsilon C}) =& -\frac{t}{RC} \\ q(t) =& \epsilon C + (q_0 - \epsilon c)e^{-t/RC} \\ =& q_0 e^{-t/RC} + \epsilon C(1 - e^{-t/RC}) \end{aligned}$$

but $\epsilon C = q_{max}$. Then

$$q(t) = q_0 e^{-t/RC} + q_{max}(1 - e^{-t/RC})$$

also

$$I(t) = \frac{dq(t)}{dt} = \frac{q_0}{RC}e^{-t/RC} + \frac{q_{max}}{RC}e^{-t/RC}$$

but $\frac{q_{max}}{RC} = \frac{\epsilon C}{RC} = \frac{\epsilon}{R} = I_{max}$ and $\frac{q_0}{R} = \frac{V_0 C}{RC} = \frac{v_0}{R} = I_0$. Then

$$I(t) = (I_{max} - I_o)e^{-t/RC}$$

Finally

$$V(t) = \frac{q(t)}{C} = \frac{q_0}{C}e^{-t/RC} + \frac{q_{max}}{C}(1 - e^{-t/RC})$$

but $\frac{q_{max}}{C} = \frac{\epsilon C}{C} = \epsilon$ and $\frac{q_0}{c} = V_0$. Then

$$V(t) = v_0 e^{-t/RC} + \epsilon(1 - e^{-t/RC})$$

Example 4.1. Charging capacitor

Let $q_0 = 0$. Then

$$\begin{aligned} q(t) &= q_{max}(1 - e^{-t/RC}) \\ I(t) &= I_{max}e^{-t/RC} \\ V(t) &= \epsilon(1 - e^{-t/RC}) \end{aligned}$$

Example 4.2. Discharging capacitor

Let $\epsilon = 0 \longrightarrow Q_{max} = \epsilon C = 0$, $I_{max} = \epsilon/R = 0$. Then

$$\begin{aligned} q(t) &= q_0 e^{-t/RC} \\ I(t) &= -I_o e^{-t/RC} \\ V(t) &= v_0 e^{-t/RC} \end{aligned}$$

4.1.2 *RL Circuit*

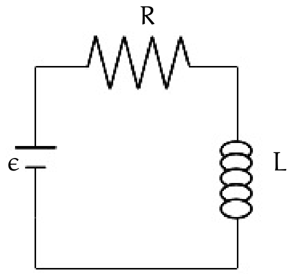


Figure 15: RL Circuit schematic diagram

When an inductor is present,

$$\oint \vec{E} \cdot \vec{ds} \neq 0$$

but rather

$$-\frac{d}{dt}\iint \vec{B} \cdot \vec{dA} = \oint \vec{E} \cdot \vec{ds}$$

Then

$$
\begin{aligned}
-L\frac{di}{dt} =& -\epsilon + iR \\
-\frac{L}{R}\frac{di}{dt} =& -\frac{\epsilon}{R} + i
\end{aligned}
$$

rearranging and integrating

$$
\begin{aligned}
\int_{i_0}^{i} \frac{di'}{i' - \epsilon/R} =& -\frac{1}{L/R}\int_0^t dt \\
\ln(\frac{i - \epsilon/R}{i_0 - \epsilon/R}) =& -\frac{t}{L/R} \\
I(t) =& \epsilon/R + (i_0 - \epsilon/R)e^{-t/L/R} \\
=& i_0 e^{-t/L/R} + \epsilon/R(1 - e^{-t/L/R})
\end{aligned}
$$

but $\epsilon/R = i_{max}$. Then

$$I(t) = q_0 e^{-t/L/R} + i_{max}(1 - e^{-t/L/R})$$

Example 4.3. Increasing current

Let $i_0 = 0$. Then

$$I(t) = I_{max}(1 - e^{-t/\tau})$$

where $\tau = L/R$

Example 4.4. Decreasing current

Let $\epsilon = 0 \longrightarrow I_{max} = \epsilon/R = 0$. Then

$$I(t) = -I_0 e^{-t/\tau}$$

where $\tau = L/R$

4.1.3 *LC Circuit*

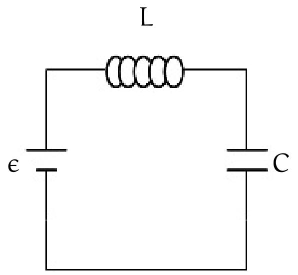


Figure 16: LC Circuit schematic diagram

The underlying principle behind the operation of an LC circuit is given by conservation of energy, i.e.

$$\frac{dU}{dt} = 0$$

Then

$$\begin{aligned}
U = U_C + U_L &= \frac{1}{2}\frac{q^2}{C} + \frac{1}{2}LI^2 \\
0 = \frac{dU}{dt} &= \frac{q}{C}\frac{dq}{dt} + LI\frac{di}{dt} \\
&= \frac{q}{C}I + LI\frac{d^2q}{dt^2} \\
&= \frac{q}{C} + L\frac{d^2q}{dt^2} \\
\frac{d^2q}{dt^2} &= -\frac{q}{LC}
\end{aligned}$$

Letting $\omega_0 = 1/\sqrt{LC}$ the solution is given by

$$q(t) = A\cos\omega_0 t + B\sin\omega_0 t$$

Because charge is conserved, we can expect that the constants A and B will shift the phase of the charge motion while keeping the charge constant. Then, let

$$\begin{aligned}
A &= q_0\cos\alpha \\
B &= -q_0\sin\alpha
\end{aligned}$$

where α is some arbitrary phase constant given by the initial conditions. Thus

$$\begin{aligned}
q(t) &= q_0\cos\omega_0 t\cos\alpha - q_0\sin\omega_0 t\sin\alpha \\
&= q_0\cos(\omega_0 t + \alpha)
\end{aligned}$$

similarly

$$\begin{aligned}
I(t) &= \omega_0 q_0\sin(\omega_0 t + \alpha) \\
&= I_0\sin(\omega_0 t + \alpha)
\end{aligned}$$

where $\omega_0 t = I_0$

Example 4.5. Initial charge

Let $q(t=0) = q_0$. Then

$$q(t=0) = q_0 = q_0\cos\alpha$$

which implies that $\alpha = 0$.

4.1.4 *LRC Circuit*

When a resistor is placed on an LC circuit energy is no longer conserved, i.e.

$$\frac{dU}{dt} \neq 0$$

but rather

$$\frac{dU}{dt} = -I^2R$$

Then

$$\begin{aligned}
U = U_C + U_L &= \frac{1}{2}\frac{q^2}{C} + \frac{1}{2}LI^2 \\
-I^2R = \frac{dU}{dt} &= \frac{q}{C}\frac{dq}{dt} + LI\frac{di}{dt} \\
-I^2R &= \frac{q}{C}I + LI\frac{d^2q}{dt^2} \\
-IR &= \frac{q}{C} + L\frac{d^2q}{dt^2} \\
0 &= \frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC}
\end{aligned}$$

letting $2\gamma = \frac{1}{L/R}$ and $\omega_0^2 = \frac{1}{LC}$ we can rewrite the differential equation to

$$0 = \frac{d^2q}{dt^2} + 2\gamma\frac{dq}{dt} + \omega_0^2 q$$

Letting $q(t) = q_0 e^{\omega_\pm t}$ and inserting into the differential equation we get

$$\begin{aligned}
\omega^2 x_0 e^{\omega_\pm t} + \gamma\omega x_0 e^{\omega_\pm t} + \omega_0^2 x_0 e^{\omega_\pm t} &= 0 \\
\omega^2 + 2\gamma\omega + \omega_0^2 &= 0
\end{aligned}$$

The roots are given by

$$\omega_\pm = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

and the solution, in general, is given by a linear combination of both solutions

$$x(t) = x_0 e^{-\gamma t}\left(c_+ e^{\sqrt{\gamma^2-\omega_0^2}t} + c_- e^{-\sqrt{\gamma^2-\omega_0^2}t}\right)$$

Letting $\delta = \sqrt{\gamma^2 - \omega_0^2}$ and $c_+ = c_- = \frac{1}{2}$, the following cases apply

- underdamped if $\delta^2 < 0 \rightarrow x(t) = x_0 e^{-\gamma t}\sin\delta t$
- overdamped if $\delta^2 > 0 \rightarrow x(t) = x_0 e^{-\gamma t}\sinh\delta t$
- critically damped if $\delta^2 = 0 \rightarrow x(t) = x_0 e^{-\gamma t}$

4.2 The Impedance Model

The analysis of circuits where the source is time-dependent can be greatly simplified with the used of complex analysis and the impedance model.

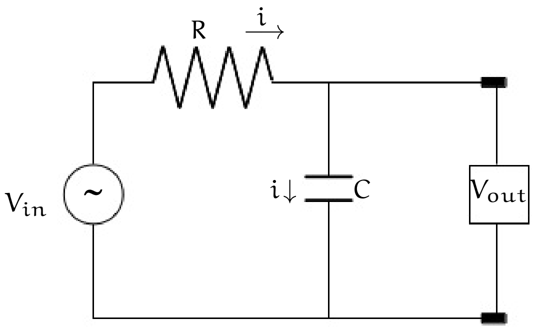


Figure 17: RC Circuit with AC source schematic diagram w

Example 4.6. RC Time-Dependent Circuit

Consider an input voltage given by $v_{in} = V_0 e^{j\omega t}$. Because the input is time- dependent, we expect the voltage across the capacitor (and the resistor) to be time-dependent as well. Then, let us denote such voltage by $v_c = V_1 e^{j(\omega t+\delta)}$, where δ is a possible phase difference with the source. Following the same approach as in the time-independent RC circuit, and assuming that the measuring device (see Figure 20a) has a large enough resistance so that negligible current goes through it, we let

$$\begin{aligned} V_R + V_C &= V_{in} \\ Ri(t) + V_1 e^{j(\omega t+\delta)} &= V_0 e^{j\omega t} \end{aligned}$$

but $C = q(t)/v(t)$, therefore $i(t) = \frac{dq(t)}{dt} = C\frac{dv(t)}{dt}$. Then

$$\begin{aligned} j\omega RCV_1 e^{j(\omega t+\delta)} + V_1 e^{j(\omega t+\delta)} &= V_0 e^{j\omega t} \\ V_1 e^{j(\omega t+\delta)}(j\omega RC + 1) &= V_0 e^{j\omega t} \end{aligned}$$

$$V_1 = V_0 \frac{1}{1 + j\omega RC} e^{-j\delta}$$

the "actual" voltage is given by the magnitude of the complex voltage. In particular, we are interested in the ratio

$$\begin{aligned} \left|\frac{V_1}{V_0}\right| &= \left|\frac{1}{1 + j\omega RC}\right| \\ &= \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}} \end{aligned}$$

We can analyze (see Figure 18) the circuit by considering the limiting cases:

- $\lim_{\omega \to 0} |v_1/v_0| = 1$
- $\lim_{\omega \to \infty} |v_1/v_0| = \frac{1}{\omega RC}$
- $\lim_{\omega \to \frac{1}{RC}} |v_1/v_0| = \frac{1}{\sqrt{2}}$ **Note:** $\omega = \frac{1}{RC}$ is called the *break frequency*

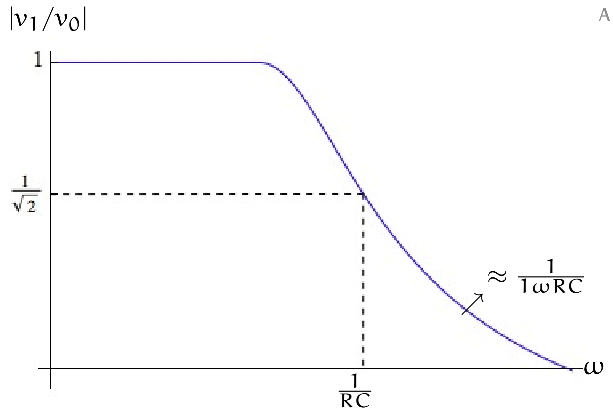


Figure 18: AC Circuit with an AC source plot

4.2.1 *The Transfer Function*

Observe that

$$\frac{1}{1+j\omega RC} \equiv \frac{1/j\omega C}{1/j\omega C+R}$$

It can also be shown (you can work this out yourself) that if in the previous example the voltage had been measured through the resistor R instead of C, i.e. $V_{out} = V_R$, the resulting relation would have been

$$\left|\frac{V_{1_R}}{V_0}\right| = \left|\frac{R}{1/j\omega C+R}\right|$$

this provides motivation for the following definitions.

Definition 4.1. Electrical Impedance

Electrical impedance Z is the measure of the opposition that a circuit presents to a current when an alternating voltage is applied. In quantitative terms, it is the complex ratio of the voltage $V(t)$ to the current $I(t)$, i.e.

$$Z = \frac{V(t)}{I(t)}$$

For a circuit with impedances $Z_1, Z_2, ...$, the total impedance can be obtained by

$$Z_T = \sum_i Z_i \quad \text{(For circuits in series)}$$

$$Z_T = \sum_i \frac{1}{Z_i} \quad \text{(For circuits in parallel)}$$

Definition 4.2. Transfer Function

We can define a transfer function $H(j\omega)$ that represents, in terms of spatial or temporal frequency, the relation between the input and output of a linear time-invariant system. The magnitude of the transfer function may be represented as

$$\boxed{|H(j\omega)| = \frac{Z_{out}}{Z_{total}}} \tag{19}$$

Example 4.7. Impedance Z_C

The impedance of a capacitor may be easily derive the following way. Suppose the capacitor has a current $i_c(t) = I_C e^{j\omega t}$ and a voltage $v_c(t) = V_C e^{j\omega t}$. Recall that for capacitors

$$\begin{aligned} i_c(t) &= C\frac{dv_c(t)}{dt} \\ I_C e^{j\omega t} &= j\omega C V_C e^{j\omega t} \\ I_C &= j\omega C V_C \end{aligned}$$

therefore

$$Z_C = \frac{V_C}{I_C} = \frac{1}{j\omega C}$$

The impedances for some common components are shown in Figure 19.

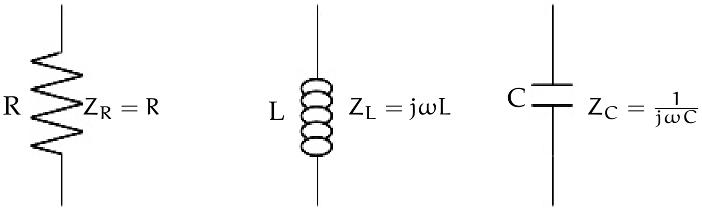
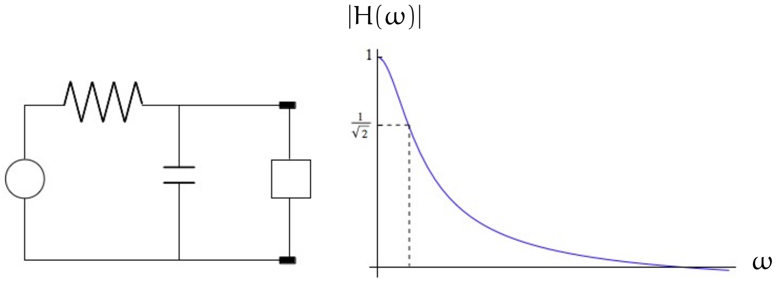


Figure 19: Complex impedances

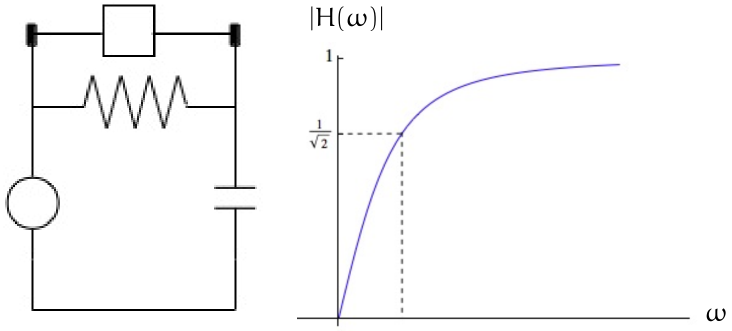
Figures 20 and 21 show a summary of the RC and RL circuits. These plots can be simply understood the following way (see Figure 19):

- For small ω, Z_C becomes huge and it acts as an open (infinite resistance). Then we expect that if we measure across the resistor no voltage difference (potential) would be measure. Conversely, for large ω, Z_C becomes very small and it acts as a short (zero resistance). Then we expect that if we measure across the resistor we would have a one-to-one relation without any voltage drop. These effects are seen in Figure 20d.

- For small ω, Z_C becomes huge and it acts as an open (infinite resistance). Then we expect that if we measure across the capacitor all voltage the difference or drop would be measure. Conversely, for large ω, Z_C becomes very small and it acts as a short (zero resistance), therefore no voltage drop would be measured. These effects are seen in Figure 20b.



(a) V_C Output. **(b)** V_C Output plot.



(c) V_R Output. **(d)** V_R Output plot.

Figure 20: RC Circuit summary.

- For large ω, Z_L becomes huge and it acts as an open (infinite resistance). Then we expect that if we measure across the resistor no voltage difference (potential) would be measure. Conversely, for small ω, Z_L becomes very small and it acts as a short (zero resistance). Then we expect that if we measure across the resistor we would have a one-to-one relation without any voltage drop. These effects are seen in Figure 20d.

- For large ω, Z_L becomes huge and it acts as an open (infinite resistance). Then we expect that if we measure across the inductor all voltage the difference or drop would be measure. Conversely, for small ω, Z_L becomes very small and it acts as a short (zero resistance), therefore no voltage drop would be measured. These effects are seen in Figure 21b.

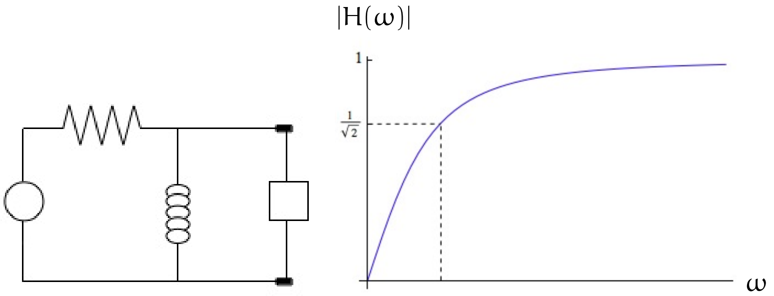
4.2.2 *Filters*

Consider Figures 20d and 21d. In both cases we measure the voltage output across the resistor (Figures 20c and 21c). What would happen if we combined the two plots?

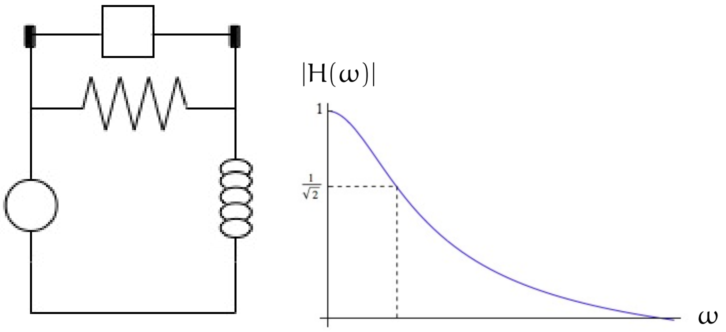
Example 4.8. RLC Circuit

Consider an RLC circuit with an AC source as shown of FIgure 22.

Suppose that we want to measure the output across the resistor. According to Definition 4.1, we can get the total impedance by adding the individual impedances for circuits in series, i.e.



(a) V_L Output. (b) V_L Output plot.



(c) V_R Output. (d) V_R Output plot.

Figure 21: RL Circuit summary.

$$\begin{aligned} Z_T &= Z_R + Z_C + Z_L \\ &= R + \frac{1}{j\omega C} + j\omega L \end{aligned}$$

and, according to Definition 19, the transfer function is then given by

$$\begin{aligned} H(j\omega) &= \frac{R}{R + \frac{1}{j\omega C} + j\omega L} \\ &= \frac{R}{R - \frac{j}{\omega C} + j\omega L} \\ &= \frac{\omega RC}{\omega RC - j + j\omega^2 CL} \\ &= \frac{\omega RC}{\omega RC - j(1 - \omega^2 CL)} \end{aligned}$$

therefore

$$|H(\omega)| = \frac{\omega RC}{\sqrt{(\omega RC)^2 + (1 - \omega^2 CL)^2}}$$

It's useful to recognize the shape of $|H(\omega)|$ quickly. The following method provides a useful approach for doing so:

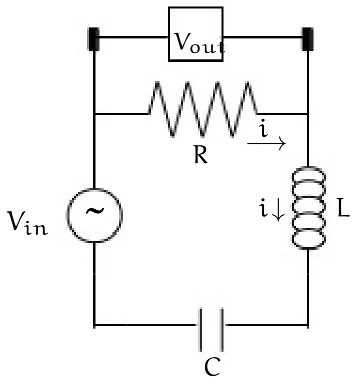


Figure 22: RLC Circuit

- as $\omega \to \infty$

$$\begin{aligned}(1-\omega^2CL)^2 &\to \omega^4C^2L^2\\(\omega RC)^2+(1-\omega^2CL)^2 &\to (\omega RC)^2+\omega^4C^2L^2\\&\to \omega^4C^2L^2\end{aligned}$$

therefore

$$|H(j\omega)| \to \frac{R}{\omega L} \quad \text{unit power decay}$$

- as $\omega \to 0$

$$\begin{aligned}(1-\omega^2CL)^2 &\to 1\\(\omega RC)^2+(1-\omega^2CL)^2 &\to 0+1\end{aligned}$$

therefore

$$|H(j\omega)| \to \omega RC \quad \text{linear decay}$$

- Finally, you may verify that as $\omega \to \frac{1}{\sqrt{LC}}$

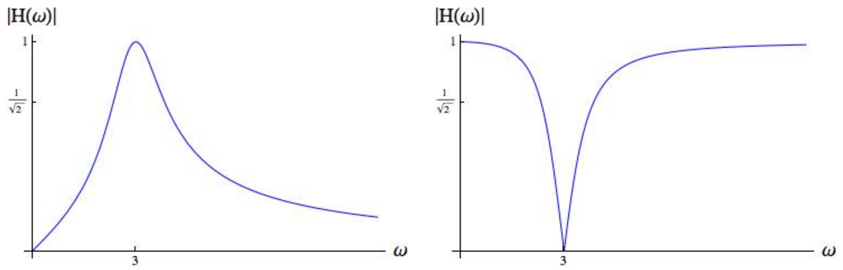
$$|H(j\omega)| \to 1$$

Conversely, we would expect the opposite outcome if we had measured the voltage across the capacitor and inductor. In particular, the resonance frequency ω_0 occurs when the impedance effects of the capacitor and inductor cancel each other, i.e.

$$\begin{aligned}0 &= Z_C+Z_L\\&=\frac{1}{j\omega C}+j\omega L\\&=-\frac{j}{\omega C}+j\omega L\\&=-1+\omega^2CL\end{aligned}$$

therefore

$$\omega_0=\frac{1}{\sqrt{LC}}$$



(a) RLC Band-pass filter.

(b) RLC Band-stop filter.

Figure 23: RLC Filter.

Let $L = 1/3$ H, $C = 1/3$ F and $R = 1/2\,\Omega$. This implies a resonance frequency of $\omega_0 = 3\,rad/s$. Figure 23 shows the plot of $|H(\omega)|$ for an output taken across the resistor (Figure 23a) and an output taken across both inductor and capacitor (Figure 23b). These are refer to as band-pass and band-stop filters. Other combinations may yield low-pass and high-pass filters (see Figures 20 and 21).

4.2.3 *Quality Factor Q*

Definition 4.3. Quality factor Q

The quality factor Q is defines as

$$Q = \frac{\omega_0}{\Delta\omega}$$

where ω_0 is the resonance frequency and $\Delta\omega$ is the band width at $|H(\omega)| = 1/\sqrt{2}$.

In general, the higher the quality factor Q, the more selective the circuit.

Example 4.9. Quality factor Q on an RLC circuit

We have seen that in an RLC circuit, when the output is taken across the resistance:

$$|H(\omega)| = \frac{\omega RC}{\sqrt{(\omega RC)^2 + (1 - \omega^2 CL)^2}}$$

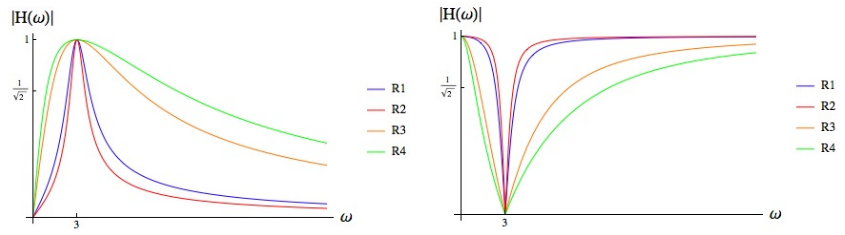
then

$$\begin{aligned} H(\omega) = \pm\frac{1}{\sqrt{2}} &= \frac{\omega RC}{\sqrt{(\omega RC)^2 + (1 - \omega^2 CL)^2}} \\ &= \frac{1}{\sqrt{(1 + (\frac{1}{\omega RC} - \frac{\omega L}{R})^2}} \end{aligned}$$

this implies

$$\begin{aligned} \pm 1 &= \frac{1}{\omega RC} - \frac{\omega L}{R} \\ 0 &= \omega^2 \mp \omega\frac{R}{L} - \frac{1}{CL} \end{aligned}$$

the roots are given by



(a) RLC Band-pass filter.
(b) RLC Band-stop filter.

Figure 24: RLC Filter with different Q values.

$$\omega_1 = -\frac{R}{2L} - \frac{1}{2}\sqrt{\frac{R^2}{L^2} - 4\frac{1}{CL}}$$
$$\omega_2 = +\frac{R}{2L} - \frac{1}{2}\sqrt{\frac{R^2}{L^2} - 4\frac{1}{CL}}$$

therefore

$$\Delta\omega = \omega_2 - \omega_1 = \frac{R}{L}$$

Also, recall that $\omega_0 = 1/\sqrt{CL}$. Then

$$Q = \frac{\omega_0}{\Delta\omega} = \frac{1}{\sqrt{CL}}\frac{L}{R} = \sqrt{\frac{L}{C}}\frac{1}{R}$$

thus, the lower the resistance, the sharper the peak

Figure 24 shows an RLC circuit with resistors $R_1 = \frac{1}{2}\,\Omega$, $R_2 = \frac{1}{3}\,\Omega$,$R_3 = 2\,\Omega$, and $R_4 = 3:\Omega$.

4.3 Experiment

A Proto Board PB-503 is available. You can generate a signal through the right TTL pins (the left are used for the ground). There are two outputs for the Oscilloscope signal. Use the 'Pin' for the signal and 'shell' for the ground. There are vertical lines of pins available, which are all on the same level for each individual vertical line. There are also horizontal sets of pins on the same level.

4.3.1 *Tasks*

- Build a high-pass filter and measure $|H(\omega)|$ as function of frequency. Measure and report the resonance frequency and the quality factor. Compared these values to the theoretical expectations and repeat for different quality factors.
- Build a low-pass filter and measure $|H(\omega)|$ as function of frequency. Measure and report the resonance frequency and the quality factor. Compared these values to the theoretical expectations and repeat for different quality factors.

- Build a band-pass filter and measure $|H(\omega)|$ as function of frequency. Measure and report the resonance frequency and the quality factor. Compared these values to the theoretical expectations and repeat for different quality factors.
- Build a band-stop filter and measure $|H(\omega)|$ as function of frequency. Measure and report the resonance frequency and the quality factor. Compared these values to the theoretical expectations and repeat for different quality factors.
- Measure the characteristics I(V) of a diode. You need to think about the design of this experiment before you do it. You may be able to use the math (+-) option of the oscilloscope.

When choosing the components (inductors, resistors, or capacitors), ensure that their values make sense. Work out the mathematics of the circuit with different values before doing the experiment. Know what to expect.

4.3.2 *Preparation key words*

RC circuits and their mathematical description, complex numbers, phase shift, measurement of currents and voltages, differential equations describing electric circuits (and their solutions in the presence of a periodic input signal and in the case of off-on DC input), Physics of a pn-junctions, diode characteristics, principle of a transistor.

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[2] Wolfson and Pasachoff *Physics* 3rd ed. Reading, MA: Addison Wesley, 1999.

5 NUCLEAR SPECTROSCOPY

The first information about the atomic nucleus came with the discovery of radioactivity by A. H. Becquerel in 1896. The rays emitted by radioactive nuclei were first classified by Rutherford as alpha α, beta β, and gamma γ, according to their ability to penetrate matter and to ionize air: α radiation (^{4}He nucleus) penetrates the least and produces the most ionization, γ (electromagnetic) radiation penetrates the most with the least ionization, and β radiation ($e^{\pm}$) is intermediate between them.

5.1 Radioactivity

In 1900 Rutherford discovered that the rate of emission of radiation from a substance was not constant but decreased exponentially with time. This exponential time dependence is characteristic of all radioactivity and indicates that it is a statistical process.

For a statistical decay (in which the decay of any individual nucleus is a random event), the number of nuclei decaying in a time interval dt is proportional to dt and to the number of nuclei present. If $N(t)$ is the number or radioactive nuclei at time t and $-dN$ is he number that decay in dt, we have

$$-dN = \lambda N dt$$

where λ is a proportionality constant called the decay constant. The solution of this differential equation is given by

$$N(t) = N_0 e^{-\lambda t} \tag{20}$$

Definition 5.1. The **decay rate** of a radioactive process is defined as

$$R = -\frac{dN}{dt} = \lambda N_0 e^{-\lambda t} = R_0 e^{-\lambda t} \tag{21}$$

Note that both the number of nuclei and the rate of decay decrease exponetially with time. It is the decrease in the rate of decay that is determined experimentally.

Definition 5.2. The **mean life time** of a radioactive process is define as

$$\tau = \int_0^\infty t f(t) dt \tag{22}$$

where $f(t)dt$ is the fraction of nuclei with lifetimes between t and $t + dt$.

Problem 5.1. Calculate the mean life time of a radioactive process with decay constant λ and a number of nuclei N.

Solution: The number of nuclei with lifetimes between t and $t + dt$ is the number that decay in dt, which is $\lambda N dt$; thus the fraction with lifetimes in dt is

$$f(t)dt = \frac{\lambda N dt}{N_0} = \lambda e^{-\lambda t} dt$$

from Equation 22, the mean life time is given by

$$\tau = \int_0^\infty t f(t) dt = \int_0^\infty t \lambda e^{-\lambda t} dt = \frac{1}{\lambda}$$

which is the reciprocal of the decay constant λ.

Definition 5.3. The **half life** $t_{1/2}$ is defined as the time after which the number of radioactive nuclei has decrease to half its original value.

Problem 5.2. Find the half life of a radioactive process with a mean life time τ.

Solution: From Equation 20

$$\frac{1}{2}N_0 = N_0 e^{-\lambda t_{1/2}} \rightarrow e^{\lambda t_{1/2}} = 2$$

this implies

$$t_{1/2} = \frac{\ln 2}{\lambda} = 0.693\tau$$

5.2 Radiation Sources

Radiation sources can be conveniently classified into four categories:

- Charged particulate radiation
  - Fast electrons ($\beta^{\pm}$ particles emitted in nuclear decay, as well as energetic electrons produced by any other process)
  - Heavy charged particles (all energetic ions with $M > 1amu$)
- Uncharged radiation
  - Electromagnetic radiation (x-rays and γ rays)
  - Neutrons (slow and fast neutrons generated by nuclear processes)

5.2.1 *Fast Electron Sources*

Fast electron sources include beta decay, internal conversion, and Auger electrons. The most common source of fast electrons in radiation measurements is a radioisotope that decays by beta-minus emission. The process is written schematically

$$^{A}_{Z}X \rightarrow ^{A}_{Z+1}Y + \beta^{-} + \overline{\nu} \tag{23}$$

where X and Y are the initial and final nuclear species, and $\overline{\nu}$ is the antineutrino.

5.2.2 *Heavy Charged Particle Sources*

Heavy charged particle sources include alpha decay and spontaneous fission. The probability of decay is governed by the barrier penetration mechanism described in most texts on nuclear physics, and the half-life of useful sources varies from days to many thousands of years. The decay process is written schematically as

$$^{A}_{Z}X \rightarrow ^{A-4}_{Z-2}Y + ^{4}_{2}\alpha \tag{24}$$

where X and Y are the initial and final nuclear species.

5.2.3 *Sources of Electromagnetic Radiation*

Sources of electromagnetic radiation include gamma rays following beta decay, annihilation radiation, gamma rays following nuclear reactions, bremsstrahlung, characteristic x-rays and synchrotron radiation.

5.3 Scintillation Detector Principles

Definition 5.4. **Luminescence** is emission of light by a substance not resulting from heat; it is thus a form of cold body radiation.

Definition 5.5. An **scintillator** is a material that exhibits scintillation – the property of luminescence when excited by ionizing radiation.

The detection of ionizing radiation by the scintillation light produced by certain materials, as shown of FIgure 25c, is one of the oldest techniques used to analyze ionizing radiation. Images could be collected or intensity measurements could be made. Measurements were also made with the human eye observing the brightness of frequency of flashes in the scintillator. Modern detection techniques employ the use of photomultiplier tubes or photodiodes to convert the light into an electrical pulse. In general, the purpose of scintillation detectors is to produce a large light output (caused by ionizing radiation) in the visible range. There are two types of scintillators: organic and inorganic crystals. The inorganic tend to have the best light output and linearity, but with several exceptions are relatively slow in their response time. In this lab you will use the inorganic alkali halide crystal sodium iodide (NaI).

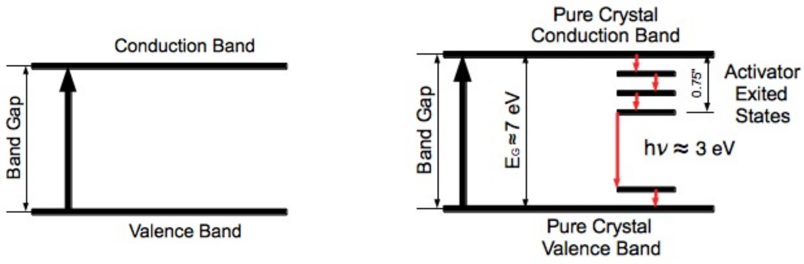
5.3.1 *Inorganic Scintillators*

The scintillation mechanism in inorganic materials depends on the energy states of the crystal structure of the material. For a given crystal structure, electrons can only exists in discrete energy levels (see Figure 25a). The valence band represents states of lower energy for electrons that are free to travel through the crystal without being bound to a specific atom, whereas the conduction band represents states at higher energy for electrons that are free to travel. For semiconductors and insulates, between these two bands is the band gap, or forbidden band, in which electrons are not allowed to exists.

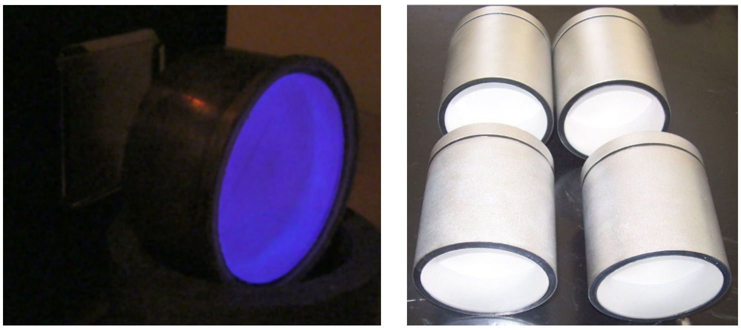
For a pure crystal, the vast majority of conduction electrons are in the valence band while the conduction band is essentially free of electrons, the energy states in the conduction band and almost all unoccupied. When an electron absorbs energy it can jump the band gap to the conduction band, leaving a hole in the normally filled valence band. The return of the electron to the valence band with a photon is an inefficient process. Furthermore, when this return does occur the resulting photon is of too high energy to lie in the visible range.

To enhance the probability of visible photon emission during the de-excitation process, small amounts of an impurity are commonly added to inorganic scintillators, a procedure commonly referred to as doping. Such deliberately added impurities, called activators, create special sites in the lattice at which the normal energy band structure is modified from that of the pure crystal. As a result, there will be energy states created within the forbidden gap through which the electron can de-excite back to the valence band.

Because the energy is less than that of the full forbidden gap, this transition can now give rise to a visible photon and therefore serve as the basis of the scintillation process. A charged particle passing through the detection



(a) Pure crystal energy band diagram. **(b)** Activated energy band diagram.



(c) Luminescence of scintillator. **(d)** Encased scintillator.

Figure 25: Sodium Iodide Thallium activated.

medium will form a large number of electron-hole pairs created by the elevation of electrons from the valence to the conduction band. The positive hole will quickly drift to the location of an activator site and ionize it, because the ionization energy of the impurity will be less than that of a typical lattice site. Meanwhile, the electron is free to migrate through the crystal and will do so until it encounters such an ionized activator. At this point the electron can drop into the activator site, creating a neutral configuration that can have its own set of excited energy states (see Figure 25b).

5.3.2 *NaI (Tl) Scintillator*

Most scintillators reported in the literature are implemented in wide-gap insulating materials doped ("activated") with radiation centers. A classical example of a solid-state scintillator is sodium iodide activated with thallium (NaI:Tl), introduced by Hofstadter[3] more than 60 years ago.

Problem 5.3. Given a 1 MeV particle energy, what is the resulting number and average energy of the photons created due to the ionizing radiation in a NaI:Tl scintillator?

Solution: A measure of the efficiency of the scintillation process follows from a simple energy calculation (see Figure 25b). For a wide category of materials, it takes on the average about three times the bandgap energy to create one electron-hole pair. In sodium iodide, this means about $20eV$ of charged particle energy must be lost to create one electron-hole pair. For $1MeV$ of particle energy deposited in the scintillator, about $5 \cdot 10^4$ electron-hole pairs are thus created. Various experimental determinations have shown that the absolute scintillation efficiency of thallium-activated sodium iodide is about 12%. Absorption of $1MeV$ of energy should therefore yield about $1.25 \cdot 10^5$ eV in total light energy, or $4 \cdot 10^4$ photons with an average energy of $3eV$. The yield is thus very close to 1 photon per electron-hole pair originally formed, and the energy transfer to activator sites must be extremely efficient.[4] [5] Typically, scintillator crystals are enclosed in a metal case as shown on Figure 25c and 25d.

The dominant decay time of the scintillation pulse is 230 ns, uncomfortably long for some fast timing or high counting rate applications. In addition to this prompt yield, a phosphorescence with characteristic 0.15 s decay time has also been measured,[7] which contributes about 9% to the overall light yield. Other longer-lived phosphorescence components have also been measured. Because the anode time constant of photomultiplier tubes is usually set much shorter than these decay times, each photoelectron associated with the phosphorescence is normally resolved individually.

5.4 Photomultiplier Tubes and Photodiodes

As previously mentioned, modern techniques require a mechanism to convert extremely low light output to a useful electrical signal. This is the purpose of the photomultiplier (PM) tube. A simplified diagram of a PM tube is shown on Figure 26. The vacuum enclosure is required to accelerate low energy electrons effectively by internal electric fields. The electron multiplier section in a PM tube provides an efficient collection geometry for the photoelectrons as well as serving as a near-ideal amplifier to greatly increase their number. After amplification through the multiplier structure, a typical scintillation pulse will give rise to $10^7 - 10^{10}$ electrons, sufficient to serve as the charge signal for the original scintillation event. This charge is conventionally collected at the anode or output stage of the multiplier structure. The PM has two main structures: the photosensitive layer (photocathode) and the electron multiplier structure.

5.4.1 *Cathode Structure*

The general purpose of the cathode structure is to convert as many photons to low energy electrons as possible. We can generalized the method by which the cathode can achieve this in three steps.

- Absorption of the photon's energy
- Migration of the electron to the surface of the cathode
- Escape of the electron from the surface of the cathode

Each of these steps must be taken into consideration when choosing the material and the materials' geometry of the cathode.

Absorption The absorption is limited by the amount of energy $h\nu$ carried by the photon.

Migration As the electrons travel towards the surface of the cathode after absorbing the photon's energy, some of this energy is lost due to collisions. The rate of energy loss in metals is relatively high, and an electron can travel no more than a few nanometers before its energy drops below the potential

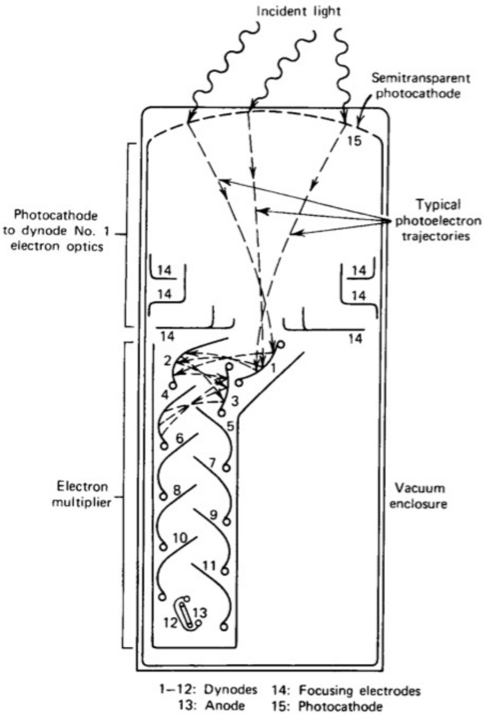


Figure 26: Basic elements of a PM tube[8]

barrier. Therefore, only the very thin layer of material lying within a few nanometers of the surface will contribute any photoelectrons from common metals. In semiconductors, the rate of energy loss is much lower and the escape depth can extend to about 25 nm. This, however, is still a very small thickness even with respect to stopping visible light. Photocathodes of this thickness are semitransparent and will cause less than half the visible light to interact within the photosensitive layer. Therefore, such photocathodes cannot come close to converting all the visible light photons into electrons, no matter how low the potential barrier may be.

Escape Finally, in the escape, there must be sufficient energy left for the electron to overcome the inherent potential barrier that always exists at any interface between material and vacuum. This potential barrier (often called the work function) is normally greater than 3 or 4 eV for most metals but can be as low as $1.5 - 2$ eV for suitably prepared semiconductors.

A unit of great significance in scintillation counting is the quantum efficiency (QE) of the photocathode. The quantum efficiency is simply defined as

$$\mathsf{QE} = \frac{\text{number of photoelectrons emitted}}{\text{number of incident photons}}$$

The quantum efficiency would be 100% for an ideal photocathode. Because of the limitations mentioned earlier, practical photocathodes show maximum quantum efficiencies of $20\% - 30\%$[1].

Problem 5.4. Given a 1 MeV particle energy, what is the resulting number and average energy of photoelectrons emitted from set up that uses a NaI:Tl scintillator and a PM tube with an efficiency of 20%?

Solution: From the previous problem, $4 \cdot 10^4$ photons are created. This implies that about $8 \cdot 10^3$ photoelectrons are emitted from the cathode. Since the average energy of the photons is 3 eV, and most of this energy is lost due to the work function, the average energy of the emitted photoelectrons is typically of 1 eV or less.

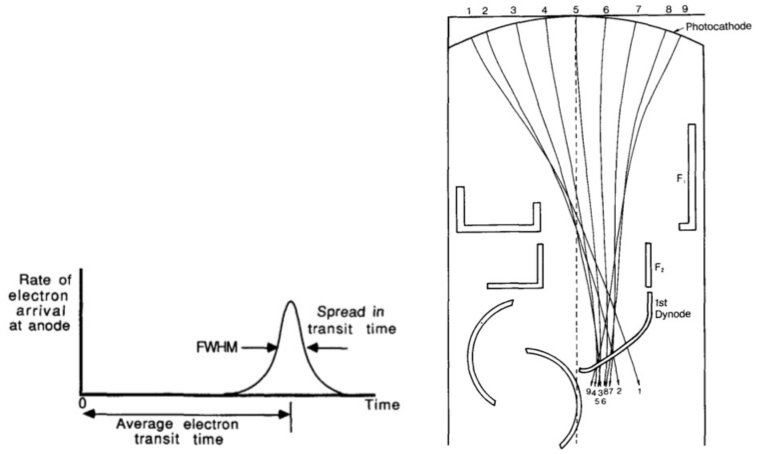
5.4.2 *Multiplier Structure*

The multiplier portion of a PM tube is based on the phenomenon of secondary electron emission. Electrons from the photocathode are accelerated and caused to strike the surface of an electrode, called a dynode. If the dynode material is properly chosen, the energy deposited by the incident electron can result in the reemission of more than one electron from the same surface. The process of secondary electron emission is similar to that of photoemission discussed in the previous section. In this case, however, electrons within the dynode material are excited by the passage of the energetic electron originally incident on the surface rather than by an optical photon.

Electrons leaving the photocathode have a kinetic energy on the order of 1 eV or less. Therefore, if the first dynode is held at a positive potential of several hundred volts, the kinetic energy of electrons on arrival at the dynode is determined almost entirely by the magnitude of the accelerating voltage. The creation of an excited electron within the dynode material requires an energy at least equal to the bandgap, which typically may be of the order of $2-3$ eV. Therefore, it is theoretically possible for one incident electron to create on the order of 30 excited electrons per 100 V of accelerating voltage. Because the direction of motion of these electrons is essentially random, many will not reach the surface before their de-excitation. Others that do arrive at the surface will have lost sufficient energy so that they cannot overcome the potential barrier at the surface and are therefore incapable of escaping. Therefore, only a small fraction of the excited electrons ultimately contribute to the secondary electron yield from the dynode surface. To achieve electron gains on the order of 10^6 all PM tubes employ multiple stages.

Because the time required for photoemission in the photocathode or secondary emission from dynodes is very short (0.1 ns or less), the time characteristics of the PM tube are determined exclusively by the electron trajectories. The electron transit time of a PM tube is defined as the average time difference between the arrival of a photon at the photo-cathode and the collection of the subsequent electron burst at the anode. In PM tubes of various designs, electron transit times range from $20-80$ ns. In most timing applications, however, the transit time itself is not of primary importance because if it were always a con- stant, it would introduce only a fixed delay in the derived signal.Instead, the spread in transit time is a more important quantity because it determines the time width of the pulse of electrons arriving at the anode of the tube. The timing response of a typical PM tube is illustrated in Figure 27a

The region between the photocathode and first dynode is critical in determining the timing properties. To allow uniform collection over large photocathodes, this distance is kept fairly large compared with interdynode distances (see Figure 27b). The difference in paths between a photoelectron



(a) The response of a PM tube to a short pulse of light on the photocathode.

(b) Electrons accelerated from the photocathode to the first dynode in a PM tube.

Figure 27: Pulse timing properties.

leaving the center of the photocathode and one at the edge is often a dominant factor in the observed spread in transit time. The photocathode is often curved to minimize the transit time spread across its diameter.

5.5 Experiment

As mentioned in the multiplier structure, the dynodes magnify the pulse of electrons, allowing them to be detected. The signal is pre-amplified and later amplified before it enters a multi channel analyzer (MCA). The analyzer separates different energies into different channels. In order to correlate the various channels to the actual energies, a sample emitting a known energy must also be used. It is known that Cesium-137 radiates at 0.662 MeV, so after locating the peak of the spectrum, this information has to be used to calibrate the MCA.

After calibrating the MCA, keep the Cesium-137 sample placed as closely as possible in front of the detector and measure the count rate for different intervals. Next, measure the absorption parameters of a 0.250-inch polyethylene filter and 0.032 and 0.062- inch lead filters. Place an Indium-115 sample in front of the detector and measured the counts at approximately 417 keV. This is a freshly activated sample. Measure the decay rate of this energy state.

5.5.1 *Tasks:*

- Make yourself familiar with the software to acquire spectra
- Measure the background signal without any radioactive source
- Measure the spectra of the radioactive sources, determine the counts per second, their statistical distribution (repeat one experiment several times), and determine the FWHM of a peak
- Measure the absorption parameters for two different filters

- Ask Raj for the neutron-activated In sample
- Measure the lifetime of the radioactive isotope (it's about 1 hour, so think about the way to measure this before you execute this experiment)

5.5.2 *Preparation key words*

Types of radiation and its detection, NaI detectors, photomultipliers (how do they work?), gamma-ray emission, Compton effect, lifetime of isotopes and its measurement, stability of isotopes and types of nuclear decay, radiation protection for different types of radiation.

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6 OPTICAL SPECTROSCOPY

The interaction between electromagnetic waves and matter can reveal properties of both EM waves and matter. **Spectroscopy** is a general term for the science that deals with the interactions of various types of radiation with matter. Some of this interactions can be approximated with classical models and others are best described with quantum mechanics, although in this manual we deal with quantum approximations as well. In this experiment you are able to measure transmitted, absorbed, and/or scattered light from certain liquids and/or foils and you will be able to identify gases by characterization of their emission spectra.

In section 8.1, *Classical Theory*, we derive the basic classical formulas that are essential for the understanding and execution of the part of the experiment that deals with transmitted, absorbed, and/or scattered light from certain liquids and/or foils. Subsection 8.1.1 presents a classical electromagnetic derivation that briefly describes the behavior of electromagnetic waves in matter. This section presents a relatively crude derivation of the complex (frequency dependent) *absorption coefficient* and *index of refraction*. For experimental purposes this section may be skipped. For students seeking a deeper understanding of EM radiation and its interaction with matter, and how such interactions relate to experimental measurements, this section can be briefly studied. Subsection 8.1.2 presents *one of the derivations* of Beer's law. Alternate, more elaborate, derivations are commonly available. This section is crucial for the understanding of this experiment.

6.1 Classical Theory

6.1.1 *Absorption Coefficient and Index of Refraction*

The material in this section is based on a discussion by Griffiths[1]. The propagation of electromagnetic waves through matter is governed by three properties of the material:

- Permittivity ϵ
- Permeability μ
- Conductivity σ

These properties depend to some extend on the frequency of the electromagnetic waves. Whenever the speed of a wave depends on its frequency, the supporting medium is called **dispersive**.

Electrons in a nonconductor are bound to specific molecules. External electromagnetic waves in a conductor disrupt the equilibrium of the system. Practically any biding force can be approximated using Taylor expansion for sufficiently small displacements from an equilibrium position x_0, e.g.

$$U(x) = U(x_0) + xU'(x_0) + \frac{1}{2}x^2U''(x_0) + \cdots$$

The first term is a constant and has no dynamic significance while the second term vanishes given that $U(x)$ has an extrema at x_o.

Letting $U''(x_0) = k_{\text{spring}}$ and $\omega_0^2 = \sqrt{\frac{k_s}{m}}$ we can approximate the biding force as

$$F_{binding} = -k_{\text{spring}}x = -m\omega_0^2 x$$

We can also approximate some sort of damping force by

$$F_{damping} = -2m\gamma\frac{dx}{dt}$$

where the factor of 2 is there only for convenience in our calculations. In the presence of an electromagnetic wave of frequency ω, polarized in the x direction, the electron is subject to a driving force

$$F_{driving} = qE = qE_0 \cos \omega t$$

Putting all this into Newton's second law gives

$$m\frac{d^2x}{dt^2} = F_{binding} + F_{damping} + F_{driving}$$

or,

$$m\ddot{x} + 2m\gamma\dot{x} + m\omega_0^2 x = qE_0 \cos \omega t$$

In the complex plane, the second order differential equation becomes

$$\ddot{\tilde{x}} + 2\gamma\dot{\tilde{x}} + \omega_0^2\tilde{x} = \frac{qE_0}{m}e^{-i\omega t}$$

Letting $\tilde{x}(t) = \tilde{x}_0 e^{-i\omega t}$

$$-\omega^2\tilde{x}_0 e^{-i\omega t} - 2i\gamma\omega\tilde{x}_0 e^{-i\omega t} + \omega_0^2\tilde{x}_0 e^{-i\omega t} = \frac{qE_0}{m}e^{-i\omega t}$$

therefore

$$\tilde{x}_0 = \frac{qE_0/m}{\omega_0^2 - \omega^2 - 2i\gamma\omega}$$

In polar notation, $\tilde{x}_0 = |\tilde{x}_0|e^{i\delta}$, then

$$\begin{aligned} |\tilde{x}_0| &= \frac{qE_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}} \\ \tan\delta &= \frac{2\gamma\omega}{\omega_0^2 - \omega^2} \end{aligned}$$

and finally

$$\tilde{x}(t) = \frac{\alpha/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}e^{-i(\omega-\delta)t} \tag{25}$$

The dipole moment is the real part of

$$\tilde{p}(t) = q\tilde{x}(t) = \frac{qE_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}e^{-i(\omega t-\delta)} \tag{26}$$

The nature of the imaginary component of Equation 26 means that p is out of phase with E–lagging behind by an angle

$$\delta = \tan^{-1}\left(\frac{2\gamma\omega}{\omega_0^2 - \omega^2}\right)$$

which is very small when when $\omega \ll \omega_0$ and rises to π when $\omega \gg \omega_0$. In general, differently situated electrons within a given molecule experience different natural frequencies and damping coefficients. Suppose that there are f_j electrons with a natural frequency ω_j and damping γ_j in each molecule. If there are N molecules per unit volume, the polarization **P** is given by the real part of

$$\tilde{\mathbf{P}} = \frac{Nq^2}{m}\left(\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - 2i\gamma_j\omega}\right)\tilde{\mathbf{E}}_0 \tag{27}$$

Now, given that $\tilde{\mathbf{P}} = \epsilon_0\tilde{\chi}_e\tilde{\mathbf{E}}$, $\tilde{\epsilon} = \epsilon_0(1+\tilde{\chi}_e)$, and $\tilde{\epsilon}_r = \frac{\tilde{\epsilon}}{\epsilon_0} = 1+\tilde{\chi}_e$ we find that the complex dilectric constant is given by

$$\tilde{\epsilon}_r = 1 + \frac{Nq^2}{m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - 2i\gamma_j\omega} \tag{28}$$

Ordinary, the imaginary term is negligible; however, when ω is very close to one fo the resonant frequencies (ω_j) it plays an important role. In a dispersive medium the wave equation for a given frequency reads

$$\frac{\partial^2\mathbf{E}}{\partial x^2} = \tilde{\epsilon}\mu_0\frac{\partial^2\mathbf{E}}{\partial t^2}$$

and it admits plane wave solutions of the form

$$\mathbf{E}(z,t) = \mathbf{E}_0 e^{i(\tilde{k}z - \omega t)} \tag{29}$$

with a complex wave number $\tilde{k} = \omega\sqrt{\tilde{\epsilon}\mu}$ with real and imaginary parts given by

$$\tilde{k} = k + i\kappa$$

which plugged into Equation 29 yield

$$\mathbf{E}(z,t) = E_0 e^{-\kappa z} e^{i(kz - \omega t)} \tag{30}$$

which implies that the wave is attenuated as the damping absorbs energy. Now, the intensity is proportional to E^2 and hence to $e^{-2\kappa z}$, then, we define

$$\alpha = 2\kappa \tag{31}$$

as the **absorption coefficient**. Furthermore, the **index of refraction** is given by

$$n = \frac{ck}{\omega} \tag{32}$$

Now, if we define the coefficient A_j by

$$A_j = \frac{f_j}{(\omega_j^2 - \omega^2)^2 + 4\gamma_j^2\omega^2} \tag{33}$$

we can re-write the complex wave number as

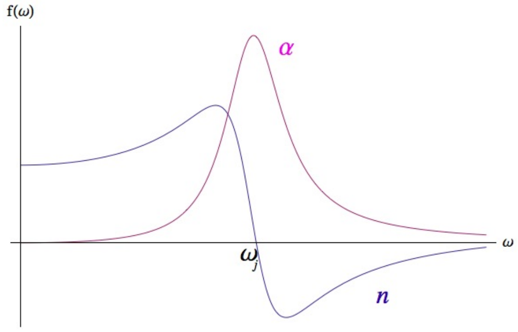


Figure 28: Anomalous Dispersion

$$\begin{aligned}\tilde{k} = \frac{\omega}{c}\sqrt{\tilde{\epsilon}_r} &= \frac{\omega}{c}\sqrt{1+\frac{Nq^2}{m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2-\omega^2-2i\gamma_j\omega}} \\ &= \frac{\omega}{c}\sqrt{\left(1+\frac{Nq^2}{m\epsilon_0}\sum_j A_j(\omega_j^2-\omega^2)\right)+2i\frac{Nq^2}{m\epsilon_0}\sum_j A_j\gamma_j\omega}\end{aligned}$$

In general, the absorption coefficient is given by

$$\boxed{\alpha = \frac{2\omega}{c}\mathrm{Re}\left[\sqrt{\left(1+\frac{Nq^2}{m\epsilon_0}\sum_j A_j(\omega_j^2-\omega^2)\right)+2i\frac{Nq^2}{m\epsilon_0}\sum_j A_j\gamma_j\omega}\right]} \tag{34}$$

Figure 28 shows the plotted index of refraction and the absorption coefficient in the vicinity of one of the resonances. As seen, the index of refraction rises gradually as the frequency increases. However, in the immediate neighborhood of a resonance the index of refraction drops sharply. This behavior is called **anomalous dispersion** and it coincides with regions of maximum absorption .

6.1.2 *Beer's Law*

The material in this section is based on a discussion by Skoog[1]. Consider a block of absorbing material (this could be solid, liquid, or gas) which in this case we take to be a solution (see Figure 29).

A beam of parallel radiation with intensity I_0 in the x direction enters the block from the left. After passing the length b of the block with n $particles$ (these could be absorbing atoms, ions, or molecules), the intensity is reduced to I. Now, consider an infinitesimal volume of thickness dx and cross-sectional area A. Within this volume there are dn absorbing particles. Now, imagine that each particle has a *capture* cross-sectional area σ at which photon capture occurs. That is, if a photon reaches this area absorbtion will occur. The sum of all capture areas is designated dA, that is, $dA = \sigma dn$; the ratio of the total capture area to the total area, then, is dA/A. On a statistical average, this ratio represents the probability for the capture of photons within each section. But if the intensity at a given section is I and dI is the reduced intensity in that section, then the probability is also denoted by $-dI/I$, with the negative sign indicating that I decreases. Then,

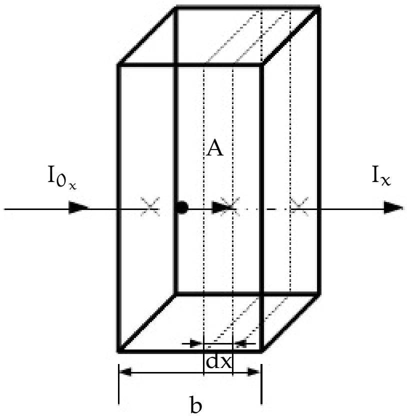


Figure 29: Attenuation of radiation with initial intensity I_0 by a solution c containing c moles per liter of absorbing solute and with a path length of b cm. $I < I_0$

$$-\frac{dI}{I} = \frac{dA}{A}$$
$$= \frac{\sigma dn}{A}$$

integrating both sides

$$-\int_{I_0}^{I} \frac{dI}{I} = \int_0^n \frac{\sigma dn}{A}$$
$$\ln \frac{I_0}{I} = \frac{\sigma n}{A}$$

converting to base 10 logarithms

$$\log \frac{I_0}{I} = \frac{\sigma n}{2.303A}$$

but the area of the section is given by $A = V/b$, therefore

$$\log \frac{I_0}{I} = \frac{\sigma b}{2.303} \frac{n}{V} \tag{35}$$

Observe that n/V has units of concentration $\frac{\#\text{particles}}{cm^3}$. This can be readily changed to units of concentration c $\frac{\#\text{moles}}{\text{Liter}}$ using the conversion factor $1000cm^3 = 1$ L and Avogadro's constant $N_A = \frac{6.02 \cdot 10^{23} \text{particles}}{\text{mol}}$, i.e.

$$c = \frac{n}{V} \frac{moles}{L} = \frac{n}{V} \frac{\cancel{particles}}{\cancel{cm^3}} \frac{1000 \cancel{cm^3}}{1\,L} \frac{mol}{6.02 \cdot 10^{23} \cancel{particles}}$$

then, in units of moles/L, the concentration of the solution is given by

$$\boxed{c = \frac{1000}{6.02 \cdot 10^{23}} \frac{n}{V} \,\text{mol/L}} \tag{36}$$

where n is the # of particles and V is the volume which contains the solution measured in cm^3 (since the factor of 1000 converts such measurement to liters). Conversely

$$\frac{n}{V}\frac{\mathit{particles}}{\mathit{cm}^3} = c\frac{\cancel{\mathit{mol}}}{\cancel{\mathit{L}}}\frac{1\,\cancel{\mathit{L}}}{1000\,\mathit{cm}^3}\frac{6.02\cdot 10^{23}\mathit{particles}}{\cancel{\mathit{mol}}}$$

feeding this onto equation 35 we get

$$\log\frac{I_0}{I} = \frac{6.02\cdot 10^{23}}{1000 * 2.303}\sigma bc$$

Defining $\epsilon = \frac{6.02\cdot 10^{23}}{1000 * 2.303}\sigma$ as the *molar extinction coefficient* we get

$$\log\frac{I_0}{I} = \epsilon bc$$

Define A as an experimental absorbance that closely approximates the true absorbance and it's obtained with the equation

$$A = \log_{10}\frac{I_0}{I}$$

Finally, the *experimental* absorbance, as opposed to the *true* absorbance, such as the one given by Equation 34, is given by

$$\boxed{A = \epsilon bc} \tag{37}$$

An experimental relationship between the *absorption coefficient*, as described in subsection 8.1.1, and the *molar extinction coefficient* is given by

$$\alpha = 2\kappa = 2\frac{2\pi}{\lambda}\epsilon$$

6.2 Quantum Theory

The characteristic spectral lines in a spectrum are due to the quantum nature of energy levels of an excited atom. When an atom gets excited the electron jumps to the next energy level. If the electron returns to a lower energy level the excess energy is released in the form of photons. These photons have a certain wavelength associated with this change in energy.

$$\lambda = \frac{hc}{E_1 - E_2} \tag{38}$$

Here λ is the wavelength, h and c are constants and the difference of E is the change in energy. For atoms there are unique energy differences correlated to the wavelength of light emitted. However, there is more than one energy level for each atom and henceforth more than one wavelength being emitted. This is where a prism or diffraction grating comes into separate

Figure 30: Optical emission spectrum of Helium

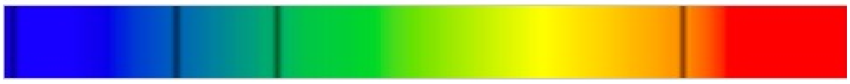


Figure 31: Optical absorption spectrum of Helium

the light into its unique wavelengths and making spectrum analysis possible. The emitting material is defined by its unique spectral lines. In Figure 30 the spectrum of helium below is shown. Each of the spectral lines coincide with excited energy level of the helium atom.

This is analogous to absorption lines. If you have a source light that emits a continuous spectra with an elemental gas in between you and the source, the gas will absorb any light that excites the atoms within that gas. Because the photon that gets emitted does not necessarily go in the direction of the observation you are left with empty sections of the spectrum shown in the figure below.

6.3 Experiment

A sample experiment with instructions and results is given. The sample provides a hands-on familiarization with the equipment and it allows students a direct application of the discussed theory.

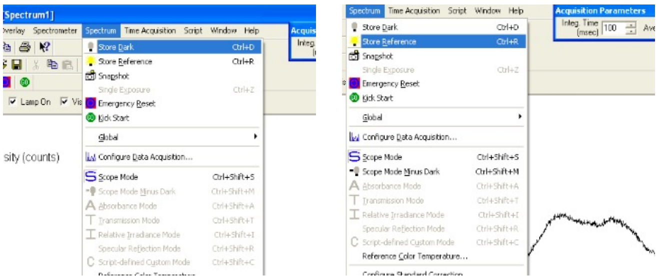
6.3.1 *Introduction: Gold Colloids*

Suppose that we have a batch of a gold colloid with a particle concentration n/V of $7.00 \cdot 10^{11}$ $\mathrm{particles/mL}$ and a particle size of 20 nm. Because the conversion factor $1000\,\mathrm{mL} = 1\,\mathrm{L}$ is equivalent to $1000\,\mathrm{cm}^3 = 1\,\mathrm{L}$, Equation 36 is still valid and this implies a concentration of $c = 1.2 \cdot 10^{-9}$ $\mathrm{mol/L}$, or $c{=}1.2 \cdot 10^{-9}$ M with $M = \mathrm{mol/L}$. Assuming a path length of $b = 1.0\,\mathrm{cm}$ we find that

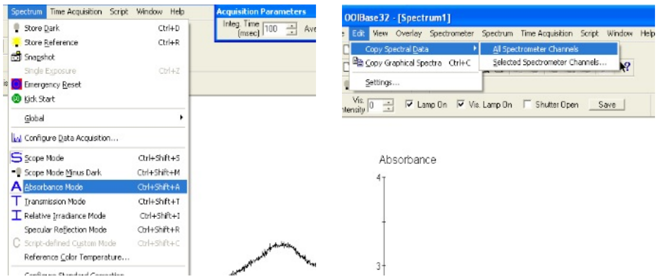
$$A = 1.2 \cdot 10^{-}9\,\epsilon\,(M\,cm) \tag{39}$$

We want to find the extinction coefficient ϵ at the wavelength of maximum of absorbance. Then:

- Turn on the light source. Open the software system.
- Place an empty sample block (with a path length b of $1\,\mathrm{cm}$) on the device (Figure 33**i**). Ensure the smooth/clear side faces the light.
- Cover the device and shift the cover all the way to the left (Figure 33**a**) such that the intensity drops to zero.
- Store the dark reference on the computer (Figure 32a).
- Shift the cover all the way to the right (Figure 33**b**). Adjust the source or the integration time to obtain an intensity of about (but not above) 4000. Store the reference (Figure 32b).



(a) Store dark. (b) Store reference.



(c) Absorbance mode for sample. (d) Store data.

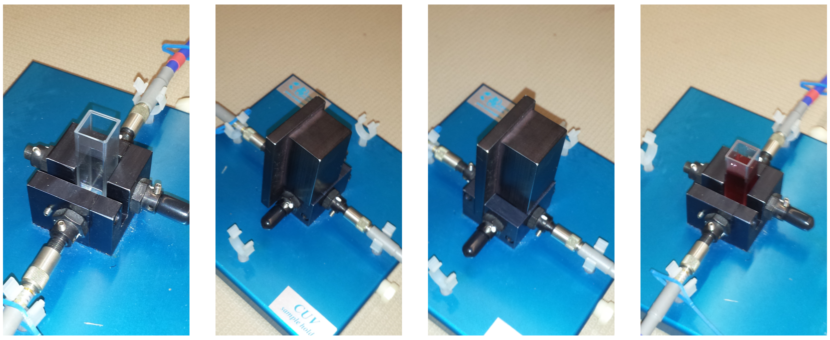
Figure 32: Software instructions.

- Place the sample on the device (Figure 33**c**) and place the cover on it. Switch the system to absorbance mode (Figure 32c).
- Store the data (Figure 32d).
- Open excel and paste the data. Label the first column as the wavelength and the second column as the absorbance.
- Using a graphing software (preferrably Origin, MATLAB, or Mathematica, although exel is *acepptable*), solve for the extinction coefficient ϵ using Equation 39.
- Graph ϵ as a function of wavelength (Figure **??**).
- From the plot, The extinction coefficient for the *peak wavelength* 523 nm is approximately $1.2 \cdot 10^9$ M^{-}cm^{-}.

6.3.2 *Tasks*

An Ocean Optics spectrometer is available for the experiment. You can use a Tungsten halogen lamp for transmission and scattering.experiments with the CUV sample holder.

- Measure the emission spectra of the different lamps available.
- Measure the characteristic maxima of the room lights.
- Measure the transmission through several foils and through selected liquids with the sample holder.
- Measure the scattered signal from liquids with the sample holder.



(i) Empty sample. (a) Dark. (b) Reference. (c) Sample.

Figure 33: Experimental set up

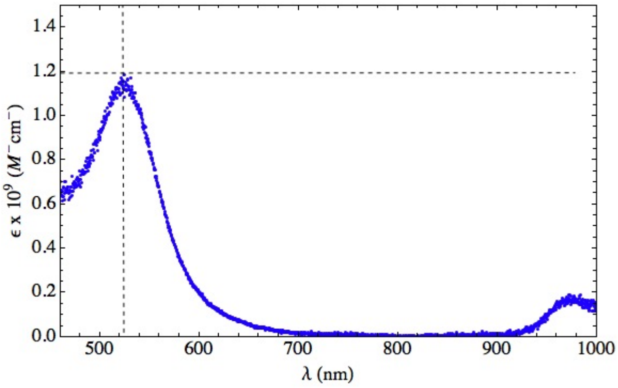


Figure 34: Extinction coeffcient of 20 nm gold colloid

6.3.3 *Preparation key words*

Lasers, characteristic emission lines from gasses, absorption, diffraction, interference, coherence, diffraction grating, calculation of interference maxima and minima for gratings and for a single slit, prism spectrometer, dispersion, refractive index.

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7 ELECTRON SPIN RESONANCE

7.1 Introduction to Quantum Mechanics

The general Shrodinger equation is given by

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi = H\Psi$$

where H is the Hamiltonian:

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V$$

In general, we want to solve this partial differential equation to obtain Ψ. For example, the standard Hamiltonian for an electron is given by

$$H = -\frac{\hbar^2}{2\mu}\Big(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\mathbf{L}^2}{\hbar^2 r^2}\Big) - \frac{Ze^2}{4\pi\epsilon_0 r}$$

The function Ψ that satisfies this differential equation $i\hbar\frac{\partial\Psi}{\partial t}\Psi = H\Psi$ also determines the orbitals of the electron around the nucleus, the possible energy states, and in general anything that can be physically measured. However, the above Hamiltonian may change in the presence of a magnetic field. As a first-order approximation, one may just simply account the added potential U by letting $H = U$ and solving the differential equation as if only the potential U was present.

7.1.1 *The Wave Function and Quantum Numbers*

A quantum mechanical system is *fully described* by its wave function Ψ. For an electron, its wave function may be dependent on the quantum numbers n, l, j, m_l, m_s, m_j, i.e. $\Psi = \Psi_{nljm_lm_sm_j}$. In Dirac notation, the wave function can be conveniently expressed as $|\Psi >= |nljm_lm_sm_j >$Wave functions are normalized such that their inner produc satisfies the relation

$$\langle\Psi||\Psi\rangle = 1$$

When we say that the system is fully described by the wave function, we mean that the wave function contains all the necessary information that can be verified through experiments. Such information include angular momentum **L**, spin angular momentum **S**, and total angular momentum **J** among others. These are related to (or rather quantized by) the quantum numbers through the relations

$$\mathbf{L}^2 = \hbar l(l+1)$$
$$\mathbf{S}^2 = \hbar s(s+1)$$
$$\mathbf{J}^2 = \hbar j(j+1)$$

Also, note that the total angular momentum **J** is given by

$$\mathbf{J} = \mathbf{L} + \mathbf{S}$$

or

$$\mathbf{L} = \mathbf{J} - \mathbf{S}$$

therefore

$$\begin{aligned}\mathbf{L}^2 &= \mathbf{J}^2 + \mathbf{S}^2 - 2\mathbf{S}\cdot\mathbf{J} \\ &= \hbar j(j+1) + \hbar s(s+1) - 2\mathbf{S}\cdot\mathbf{J}\end{aligned}$$

Consequently

$$\mathbf{S}\cdot\mathbf{J} = \frac{1}{2}\big(\hbar j(j+1) + \hbar s(s+1) - \hbar l(l+1)\big) \tag{40}$$

The quantum numbers have the following properties:

$$\begin{aligned}\text{Principal} \quad & n \longrightarrow 1 \leqslant n \\ \text{Azimuthal} \quad & l \longrightarrow 0 \leqslant l \leqslant n-1 \\ \text{Spin} \quad & s \\ \text{Total angular momentum} \quad & j \longrightarrow j = |l \pm s| \\ \text{Magnetic} \quad & m_l \longrightarrow -l \leqslant m_l \leqslant l \\ \text{Spin projection} \quad & m_s \longrightarrow -s \leqslant m_s \leqslant s \\ \text{Total angular momentum projection} \quad & m_j \longrightarrow m_j = m_l + m_s\end{aligned}$$

Most particles that we experience in everyday life have spin $s = 1/2$. These particles are *electrons*, *protons*, *neutrons*, *neutrinos*, and *muons*.

7.1.2 *Operators*

In linear algebra terms, an operator is a matrix that obeys the eigenvalue equation

$$A\vec{x} = \lambda_A \vec{x}$$

where we say that λ_A is an eigenvalue of the matrix A. In quantum mehcanics, physical observables orrespond to such eigenvalues. That is, an operator acts on the wave function to obtain physically allowed values that correspond to such operator.

Example 7.1. Momentum

The momentum operator is given by $\hat{p} = -i\hbar\frac{\partial}{\partial x}$. Suppose that the wave function is given by $|\Psi >= e^{i(kx-\omega t)}$. Then

$$\begin{aligned}\hat{p}|\Psi > &= -i\hbar\frac{\partial}{\partial x}e^{i(kx-\omega t)} \\ &= -i\hbar(ik)e^{i(kx-\omega t)} \\ &= \hbar k e^{i(kx-\omega t)}\end{aligned}$$

therefore

$$\hat{p}|\Psi > = \hbar k|\Psi >$$

which obeys the De Broglie relation $p = \hbar k$. Some typical operators and eigenvalues are:

$$\begin{aligned}
\hat{L}|\Psi> &= \hbar l|\Psi> \\
\hat{L}^2|\Psi> &= \hbar^2 l(l+1)|\Psi> \\
\hat{L}_z|\Psi> &= \hbar m_l|\Psi> \\
\hat{S}|\Psi> &= \hbar s|\Psi> \\
\hat{S}^2|\Psi> &= \hbar^2 s(s+1)|\Psi> \\
\hat{S}_z|\Psi> &= \hbar m_s|\Psi>
\end{aligned}$$

7.1.3 *Expectation Values*

The expectation value of an operator $\mathcal{O}$ is given by

$$< \mathcal{O} > = < \Psi|\mathcal{O}|\Psi >$$

where $< \Psi|$ is the complex conjugate of $|\Psi >$.

Example 7.2. Kinetic Energy

The kinetic energy operator is given by $\hat{K} = \frac{\hat{p}^2}{2m} = \frac{1}{2m}(-i\hbar\frac{\partial}{\partial x})(-i\hbar\frac{\partial}{\partial x}) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$. Suppose that the wave function is given by $|\Psi >= e^{i(kx-\omega t)}$. If we were to measure the kinetic energy of identical systems, we refer to the average value of such measurements as the expectation value, i.e.

$$\begin{aligned}
\hat{K}|\Psi > &= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}e^{i(kx-\omega t)} \\
&= \frac{\hbar^2 k^2}{2m}e^{i(kx-\omega t)}
\end{aligned}$$

therefore

$$\begin{aligned}
< \Psi|\hat{K}|\Psi > &= e^{-i(kx-\omega t)}\frac{\hbar^2 k^2}{2m}e^{i(kx-\omega t)} \\
&= \frac{\hbar^2 k^2}{2m}
\end{aligned}$$

thus

$$< \hat{K} > = \frac{\hbar^2 k^2}{2m}$$

7.2 Dipole Moments

Figure shows a primitive model of an electron in an atom. We picture the electron (mass m, charge $-e$ as moving in a circular orbit with radius r and speed v. This moving charge is equivalent to a current loop. Recall that the magnetic dipole moment is given by $\mu = IA$; where I is the current of the loop and A is the area of the loop.

To find the current associated with the electron, we note that the orbital period T (the time for the electron to complete one orbit) is the orbit circunference divided by the electron speed: $T = 2\pi r/v$. The equivalent current I

is the total charge passing any point on the orbit per unit time, which is just the value e of the electron charge divided by the orbital period T:

$$I = -\frac{e}{T} = -\frac{ev}{2\pi r}$$

The magnetic moment $\mu = IA$ is then

$$\begin{aligned}\mu &= -\frac{ev}{2\pi r}\pi r^2\\ &= -\frac{evr}{2} = \frac{emvr}{2m}\\ &= -\frac{e}{2m}L\end{aligned}$$

Experimentally, the electron's spin property behaves similar to that of angular momentum. Then we may express the magnetic dipole moment associated with spin as

$$\mu = -\frac{e}{m}L$$

where the extra factor of two was explained by Dirac in his relativistic theory of the electron

.

In gneral, we may then define the following:

$$\boxed{\mu_l = -\frac{e}{2m}\mathbf{L}} \tag{41}$$

$$\boxed{\mu_s = -\frac{e}{m}\mathbf{S}} \tag{42}$$

7.3 The Zeeman Effect

7.3.1 *Theory*

Recall that there is an associated potential energy for a magnetic dipole in the presence of a magnetic field: $U = -\boldsymbol{\mu} \cdot \mathbf{B}$. In an atom, this is an extra potential energy in addition to the standard Coloumb potential. We may say that if the electron was previously in equilibrium, then the presence of an electric field causes a *perturbation*.

When an atom is placed in a uniform external magnetic field **B**, the energy levels are shifted. This phenomenon is known as the **Zeeman effect**. For a single electron the perturbation is

$$\begin{aligned}H_Z &= -(\mu_l + \mu_s) \cdot \mathbf{B}\\ &= \frac{e}{2m}(\mathbf{L} + 2\mathbf{S}) \cdot \mathbf{B}\end{aligned}$$

There is an internal magnetic field in the atom. The nature of the Zeeman splitting depends critically on the strength of the external field in comparison to the internal field that gives rise to spin-orpit coupling. Here we consider the cases where $B_{ext} << B_{int}$.

The Zeeman correction to the energy is given by the expectation value

$$\begin{aligned} E_Z &=< \Psi|H_z|\Psi > \\ &= \frac{e}{2m}\langle \mathbf{L}+2\mathbf{S}\rangle \cdot \mathbf{B} \end{aligned}$$

Observe that

$$\begin{aligned} \mathbf{J} &= \mathbf{L}+\mathbf{S} = \mathbf{L}+2\mathbf{S}-\mathbf{S} \\ \mathbf{J}+\mathbf{S} &= \mathbf{L}+2\mathbf{S} \end{aligned}$$

therefore

$$E_Z = \frac{e}{2m}\langle \mathbf{J}+\mathbf{S}\rangle \cdot \mathbf{B}$$

The problem here is that in the presence of a magnetic field (or spin-orbit coupling), **L** and **S** are not separately conserved, meaning that their expectation values are time dependent. However, the total angular momentum $\mathbf{J} = \mathbf{L}+\mathbf{S}$ is constant (see Figure); **L** and **S** precess rapidly about this fixed vector. In particular, the (time) average value of **S** is just its projection along **J**:

$$\mathbf{S}_{avg} = \frac{(\mathbf{S}\cdot\mathbf{J})}{J^2}\mathbf{J}$$

but $\mathbf{S}\cdot\mathbf{J}$ is given by Equation 40, therefore

$$\begin{aligned} \langle \mathbf{J}+\mathbf{S}\rangle &= \langle \mathbf{J}+\mathbf{S}\rangle \\ &= \langle \mathbf{J}+\frac{(\mathbf{S}\cdot\mathbf{J})}{J^2}\mathbf{J}\rangle \\ &= \Big\langle (1+\frac{(\mathbf{S}\cdot\mathbf{J})}{J^2})\mathbf{J}\Big\rangle \\ &= \Big\langle (1+\frac{\frac{1}{2}\big(\hbar j(j+1)+\hbar s(s+1)-\hbar l(l+1)\big)}{\hbar j(j+1)})\mathbf{J}\Big\rangle \\ &= (1+\frac{\frac{1}{2}\big(\hbar j(j+1)+\hbar s(s+1)-\hbar l(l+1)\big)}{\hbar j(j+1)})\langle \mathbf{J}\rangle \end{aligned}$$

we may now define the following

Definition 7.1. Lande g-factor g_J

$$g = 1+\frac{\big(j(j+1)-l(l+1)+s(s+1)\big)}{2j(j+1)}$$

for an electron with s=1/2

$$g_J = 1+\frac{\big(j(j+1)-l(l+1)+\frac{3}{4}\big)}{2j(j+1)}$$

Then,

$$\langle \mathbf{J} + \mathbf{S} \rangle = g_J \langle \mathbf{J} \rangle$$

Suppose that we place the external magnetic field in the z direction so that only $\mathbf{J} = J_z$ makes any contribution, then

$$\begin{aligned}\langle \mathbf{J} + \mathbf{S} \rangle &= g_J \langle L_z \rangle \\ &= g_J \langle \Psi | L_z | \Psi \rangle \\ &= g_J \hbar m_j \langle \Psi || \Psi \rangle \\ &= g_J \hbar m_j\end{aligned}$$

Consequently

$$E_Z = \frac{e\hbar}{2m} g_J m_j B_z$$

and using the **Bohr magneton** $\mu_B = \frac{e\hbar}{2m}$ we get

$$\boxed{E_Z = \mu_B g_J m_j B_z} \tag{43}$$

7.3.2 *Free Radicals*

Recall that $j = |l \pm s|$ and $m_j = m_l + m_s$. For most organic radicals and radical ions, unpaired electrons have L close to zero and the total electron angular momentum quantum number J is pretty much the spin quantum number, S. This implies $j \approx s$ and $m_j \approx m_s$. Then

$$\begin{aligned}g &\approx 1 + \frac{s(s+s) + s(s+1)}{2s(s+1)} \\ &\approx 1 + \frac{1+1}{2} \\ &\approx 2\end{aligned}$$

and

$$E_Z \approx g \mu_B m_s B_z \tag{44}$$

where $m_s = \pm 1/2$

.

Some electrons will align parallel and others will anti-parallel to the magnetic field. This is the essense of the $m_s = \pm 1/2$ quantum number. Parallel and anti-parallel electrons have an energy difference of $\Delta E = E_+ - E_-$. Then

$$\boxed{\Delta E = g \mu_B B_z} \tag{45}$$

The situation becomes much more complicated with transition metals. Not only they have large L's and S's, but these values depend on the surrounding electric fields of ligands, making everything messier but also more interesting. If the molecule contains nuclei with magnetic moments, such as protons, their interaction with external field and the electronic magnetic moment will change stationary energies E_Z.

The nuclear angular momentum quantum number I determines the nuclear magnetic moment the same way as for the electron:

$$E_Z = \mu_N g_N m_i B_z$$

where μ_N is now the nuclear Bohr magneton $\mu_N = \frac{e\hbar}{2m_p}$ which is a much smaller values because of the ≈ 2000 times more heavy proton mass m_p.

7.4 Magnetic Resoance

There are basically two types of magnetic resonance experiments: nuclear magnetic resonance (NMR) and electron paramagnetic resonance (EPR), the latter sometimes called electron spin resonance (ESR). The experiments are basically the same, except that the Bohr magneton $\mu_B = e\hbar/2m_e$ and the nuclear magneton $\mu_N = e\hbar/2m_p$ differ by a factor of about 1,800 due to the mass differences between the electron and the proton. In the experiment, one applies a constant magnetic induction B_0 in some fixed direction (which we shall call the z direction), and then an oscillatory, circularly polarized magnetic induction of strength B_1 in the xy plane (perpendicular to the direction of the static field). Thus, the Hamiltonian is

$$\begin{aligned} H &= -\boldsymbol{\mu}_s \cdot \mathbf{B}_0 - \boldsymbol{\mu}_s \cdot \mathbf{B}_1 \\ &= \gamma B_0 S_z - \gamma B_1 [S_x \cos \omega t + S_y \sin \omega t] \end{aligned}$$

where $\gamma = g\mu_B$ or $g\mu_N$ for EPR nd NMR experiments, respectively. For simplicity we write

$$\begin{aligned} \gamma B_0 &= \omega_L \\ \gamma B_1 &= \omega_1 \end{aligned}$$

where ω_L is the Larmor frequency, and ω_1 is the effective frequency due to the amplitude of the applied oscillatory field. This is different from the actual angular frequency ω of the applied oscillatory field essential to flip the spins. Note that B_0 and B_1 are kept constant in the experiment, but the angular frequency ω of the oscillatory field is varied until resonance is found.

We neglect the orbital motion of the electron or proton and write the spin state (this is the wave function which neglets everything except spin)

$$\mathcal{X}(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$$

and the Schrodinger wave equation for the spin 1/2 particle may then be written as

$$\begin{aligned} H\mathcal{X}(t) &= i\hbar\frac{\partial\mathcal{X}(t)}{\partial t} \\ i\hbar\begin{pmatrix}\dot{a}\\ \dot{b}\end{pmatrix} &= \frac{\hbar}{2}\begin{pmatrix}\omega_L & \omega_1 e^{-i\omega t}\\ \omega_1 & -\omega_L\end{pmatrix} \end{aligned}$$

which leads to the two coupled differential equations

$$\begin{aligned} i\dot{a} &= \frac{1}{2}\omega_L a + \left(\frac{1}{2}\omega_1 e^{-i\omega t}\right) b \\ i\dot{b} &= -\frac{1}{2}\omega_L b + \left(\frac{1}{2}\omega_1 e^{i\omega t}\right) a \end{aligned}$$

Now in the absence of the oscillatory field ($\omega_1 = 0$), we would have

$$\begin{aligned} i\dot{a}_0(t) &= \frac{1}{2}\omega_L a_0(t) \\ i\dot{b}_0(t) &= -\frac{1}{2}\omega_L b_0(t) \end{aligned}$$

where the subscript 0 just means that $\omega_1 = 0$, but $a_0(t)$ and $b_0(t)$ are functions of t due to Larmor precession. Those solutions are

$$\begin{aligned} a_0(t) &= a_0(0)e^{-i\omega_L t/2} \\ b_0(t) &= b_0(0)e^{+i\omega_L t/2} \end{aligned}$$

we therefore write

$$\begin{aligned} a(t) &= A(t)e^{-i\omega_L t/2} \\ b(t) &= B(t)e^{-i\omega_L t/2} \end{aligned}$$

where $A(t)$ and $B(t)$ contain the extra t-dependence not in the Larmor precession. We expect that the oscillatory time dependence of $A(t)$ and $B(t)$ will be weak near "resonance", although we don't yet know where the resonance will occur. Thus

$$\begin{aligned} i\dot{a} &= i\dot{A}e^{-i\omega_L t/2} + \frac{1}{2}\omega_L A e^{-i\omega_L t/2} \\ &= \frac{1}{2}\omega_L A e^{-i\omega_L t/2} + \frac{1}{2}\omega_1 B e^{i(\omega_L/2-\omega)t} \\ i\dot{A}e^{-i\omega_L t/2} &= \frac{1}{2}\omega_1 B e^{i(\omega_L/2-\omega)t} \\ i\dot{b} &= i\dot{B}e^{i\omega_L t/2} - \frac{1}{2}\omega_1 A e^{i(\omega_L/2-\omega)t} \\ &= -\frac{1}{2}\omega_L B e^{i\omega_L t/2} + \frac{1}{2}A e^{i(\omega_L/2-\omega)t} \\ i\dot{B}e^{i\omega_L t/2} &= \frac{1}{2}\omega_L A e^{i(\omega_L/2-\omega)t} \end{aligned}$$

taking the derivative we have

$$\begin{aligned}
i\ddot{A}e^{-i\omega_L t/2} + \frac{1}{2}\dot{A}\omega_L e^{-i\omega_L t/2} &= \frac{1}{2}\omega_1 \dot{B}e^{i(\omega_L/2-\omega)t} \\
&+ \frac{1}{2}\omega_1[(\omega_L/2-\omega)]A^{i(\omega_L/2-\omega)t} \\
i\ddot{A}e^{-i\omega_L t/2} + \frac{1}{2}\dot{A}\omega_L e^{-i\omega_L t/2} &= \frac{1}{2}e^{-i\omega t}\frac{\omega_1}{2i}Ae^{i(\omega_L/2-\omega)t} \\
&+ \frac{1}{2}\frac{i(\omega_L/2-\omega)}{\frac{\omega_1}{2}}i\dot{A}e^{-i\omega_L t/2} \\
i\ddot{A}e^{-i\omega_L t/2} + \frac{1}{2}\dot{A}\omega_L e^{-i\omega_L t/2} &= -i(\frac{\omega_1}{2})^2Ae^{-i\omega_L t/2} - (\frac{\omega_L}{2}-\omega)\dot{A}e^{-i\omega_L t/2}
\end{aligned}$$

We note that the $e^{-i\omega_L t/2}$ factors appear in every term, and can be divided out. Simplifying, we obtain

$$\ddot{A} - i(\omega_L - \omega)\dot{A} + (\omega_1/2)^2 A = 0$$

Thus, we may solve this second order ordinary differential equation by assuming an exponential type of solution,

$$\begin{aligned}
A(t) &= A(0)e^{i\Omega t} \\
-\Omega^2 A(0)e^{i\Omega t} - i(\omega_L - \omega)e^{i\Omega t}(\omega_1/2)^2 A(0)e^{i\Omega t} &= 0 \\
-\omega^2 + (\omega_L - \omega)\Omega + (\omega_1/2)^2 &= 0
\end{aligned}$$

which is easily solved to yield the two solutions,

$$\frac{1}{2}[\omega_L - \omega \pm \sqrt{(\omega - \omega_L)^2 + \omega_1^2}] = \Omega_\pm$$

The two solutions of the second-order differential equation may then generally be written as

$$A_+ e^{i\Omega_\pm t} + A_e^{i\Omega_- t} = A(t)$$

where $A_\pm$ are as yet unspecified constants. Then

$$\begin{aligned}
i\dot{A}e^{-i\omega_L t/2} &= \frac{1}{2}\omega_1 e^{i(\omega_L/2-\omega)t} \\
B(t) &= -\frac{2}{\omega_1}\left(A_+\Omega_+ e^{-i\Omega t} + A_-\Omega_- e^{-i\Omega_+ t}\right) \\
&= -\frac{2}{\omega_1}\left(A_+\Omega_+ e^{-i\Omega_- t} + A_-\Omega_- e^{-i\Omega_+ t}\right)
\end{aligned}$$

Now, let us assume that at $t = 0$, the system is in a spin up state. Then $a(0) = A(0) = 1$ and $b(0) = B(0) = 0$, or

$$\begin{aligned}
A_+ + A_- &= 1 \\
A_+\Omega_+ + A_-\Omega_- &= 0 \\
A_+\Omega_+(1 - A_+)\Omega_- &= 0 \\
A_+ &= -\frac{\Omega_-}{\Omega_+ - \Omega_-}, \\
A_- &= 1 - A_+ = \frac{\Omega_+}{\Omega_+ - \Omega_-},
\end{aligned}$$

Now, the probability that at time t the spin has flipped to the down state is

$$\begin{aligned}
P(\omega, t) = |b(t)|^2 = |B(t)|^2 &= \frac{4}{\omega_1^2}\Big|A_+\Omega_+ e^{-i\Omega_- t} + A_-\Omega_- e^{-i\Omega_+ t}\Big|^2 \\
&= \frac{4}{\omega_1^2}\Big| - \frac{\Omega_-\Omega_+}{\Omega_+ - \Omega_-} e^{-i\Omega_- t} + \frac{\Omega_+\Omega_-}{\Omega_+ - \Omega_-} e^{-i\Omega_+ t}\Big|^2 \\
&= \frac{4}{\omega_1^2}\Big(\frac{\Omega_+\Omega_-}{\Omega_+ - \Omega_-}\Big)^2 (e^{-i\Omega_- t} - e^{-i\Omega_+ t}) \\
&= \frac{8}{\omega_1^2}\Big(\frac{\Omega_+\Omega_-}{\Omega_+ - \Omega_-}\Big)^2 \Big(1 - \cos[(\Omega_+ - \Omega_-)t]\Big) \\
&= \frac{8}{\omega_1^2}\Big(\frac{\frac{1}{4}[(\omega_L - \omega)^2 - (\omega_L - \omega)^2 - \omega_1^2]}{\sqrt{(\omega_L - \omega)^2 + \omega_1^2}} \\
&\times \Big(1 - \cos[t\sqrt{(\omega_L - \omega)^2 + \omega_1^2}]\Big) \\
&= \frac{8\omega_1^2}{16\omega_1^2}\frac{1}{(\omega_L - \omega)^2 + \omega_1^2}\Big(1 - \cos[t\sqrt{(\omega_L - \omega)^2 + \omega_1^2}]\Big) \\
&= \frac{1}{2}\omega_1^2\frac{1}{(\omega_L - \omega)^2 + \omega_1^2}\Big(1 - \cos[t\sqrt{(\omega_L - \omega)^2 + \omega_1^2}]\Big)
\end{aligned}$$

The maximum probability of flipping a spin is at resonance, when $\omega = \omega_L$ Then, the probability of flipping a spin becomes

$$P(\omega, t) = \frac{1}{2}[1 - \cos\omega_1 t]$$

The linewidth (full width at half maximum) is given by ω_1, the strength of the oscillatory field. The frequency dependence of the lineshape, given by the time average of the probability of flipping the spin is

$$\boxed{\big\langle P(\omega, t)\big\rangle_t = \frac{\omega_1^2}{2}\frac{1}{(\omega_L - \omega)^2 + \omega_1^2}} \tag{46}$$

is Lorenztian. This is the basis for all NMR uses in chemistry and solid state physics, and for MRI (magnetic resonance imaging) detectors in medical anal- ysis. In that case, one has to have a spatially non-uniform magnetic field, in order to detect the local positions of the flipped spins.

In experiments, the derivative of Equation 46 can usually be inferred by changes in the alternating magenetic field. That is, one is usually able to measure

$$\frac{\partial\big\langle P(\omega, t)\big\rangle_t}{\partial\omega} = \frac{\omega_1^2 + \omega_L - \omega}{(\omega_1^2 + (\omega_L - \omega)^2)^2}$$

Figure 35 (left) shows the probability and its derivative that the spin will flip. Experimentally, If the system is constantly sweep the resulting plot is shown below (right).

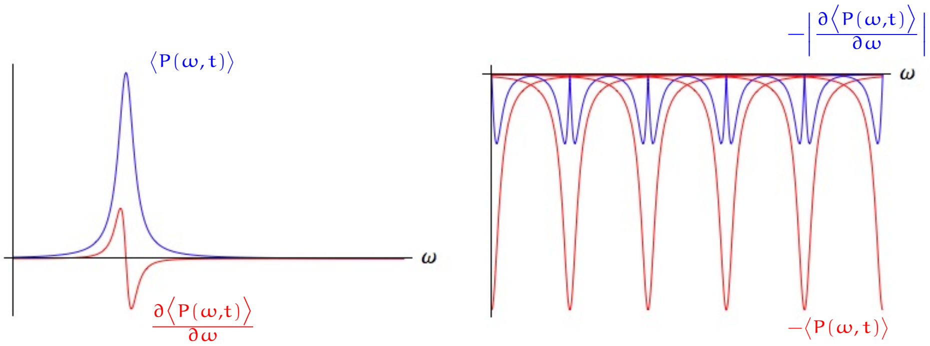

Figure 35: Probability and its derivative that the spin will flip

7.5 Experiment

The energy levels on an electron in a magnetic field split. We refer to this split as the Zeeman effect. When the angular momentum is negligible, such as in the case with radicals, the energy difference between the two split levels is given by:

$$\begin{aligned}\Delta E &= g\mu_B B_z \\ &= g\frac{e\hbar}{2m}B_z\end{aligned}$$

To test this theory, we place a constant magnetic field in the z direction and an alternating field in the xy plane perperdicular to the constant field. With this new field, the probability that an electron will absorb the radiation energy and flip its state is given by:

$$\langle P(\omega,t)\rangle_t = \frac{\omega_1^2}{2}\frac{1}{(\omega_L-\omega)^2+\omega_1^2}$$

When energy is absorbed, $\Delta E = \omega\hbar$, therefore $\omega = \frac{g\mu_B B_z}{\hbar}$, or

$$\omega = \frac{ge}{2m}B_z$$

Make yourself familiar with the setup of the equipment, how to control field, field modulation, and frequency, as well as monitoring the two latter signals on the oscilloscope. Select the largest high-frequency coil and insert one of the two samples with metal ions.

Select a high frequency, and scan through the available range of the magnetic field.

Calculate the field near the sample, and check this value with the Hall probe.

Find a field range where the ESR signal appears and lower the field modulation amplitude to identify the magnitude of the field. Determine the Bohr magneton.

Repeat the procedure for both samples at different frequencies.

7.5.1 *Preparation key words*

Principles of ESR and NMR, electrons in magnetic fields, Pauli principle, energy levels of electrons in a magnetic field, Zeeman effect, Quantum mechanical hydrogen model

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