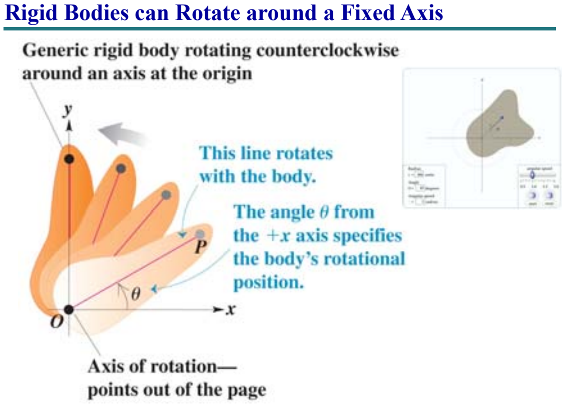
Chapter 8

Rotational Kinematics

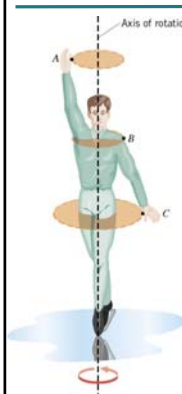
Goals for Chapter 8

- To study **angular velocity** and **angular acceleration**.
- To examine rotation with constant angular acceleration.
- To understand the **relationship between linear and angular quantities**.
- To determine the **kinetic energy of rotation** and the **moment of inertia**.
- To study rotation about a moving axis.

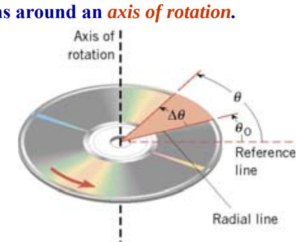
Rigid Bodies can Rotate around a Fixed Axis



Rotational Motion and Angular Displacement



In the simplest kind of rotation, points on a rigid object move on circular paths around an *axis of rotation*.



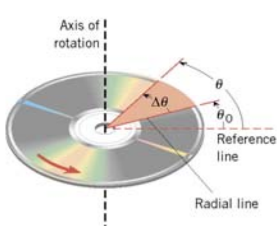
The angle through which the object rotates is called the *angular displacement.*

$$\boxed{\Delta\theta = \theta - \theta_o}$$

Definition of Angular Displacement

When a rigid body rotates about a fixed axis, the angular displacement is the angle swept out by a line passing through any point on the body and intersecting the axis of rotation perpendicularly.

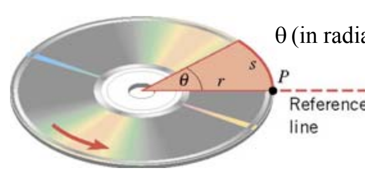
By convention, the angular displacement is positive if it is counterclockwise and negative if it is clockwise.



2 π radians = 360°

SI Unit of Angular Displacement: radian (rad)

Angular Displacement is measured in Radians



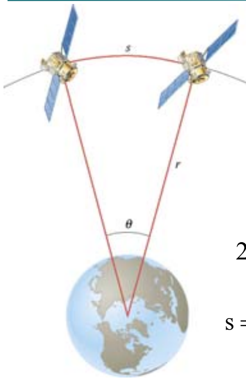
$$\theta\,(\text{in radians}) = \frac{\text{Arc length}}{\text{Radius}} = \frac{s}{r}$$

For a full revolution: $\theta = \dfrac{2\pi r}{r} = 2\pi\ \text{rad}$

$$\Rightarrow\ 2\pi\ \text{rad} = 360^\circ$$

$$1\ \text{rad} = \frac{360^\circ}{2\pi} = \frac{180^\circ}{\pi} = 57.3^\circ.$$

Example: Adjacent Synchronous Satellites



Synchronous satellites are put into an orbit whose radius is 4.23×10⁷m. If the angular separation of the two satellites is 2.00 degrees, find the arc length that separates them.

$$\theta\,(\text{in radians}) = \frac{\text{Arc length}}{\text{Radius}} = \frac{s}{r}$$

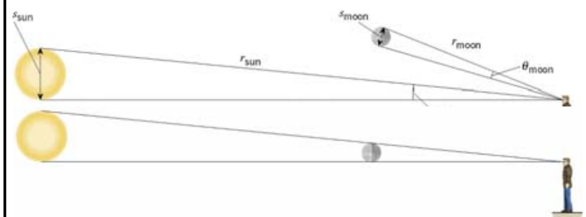
$$2.00\,\text{deg}\left(\frac{2\pi\,\text{rad}}{360\,\text{deg}}\right) = 0.0349\,\text{rad}$$

$$s = r\,\theta = \left(4.23\times10^{7}\,\text{m}\right)\left(0.0349\,\text{rad}\right)$$

$$= 1.48\times10^{6}\,\text{m}\ \ (920\,\text{miles})$$

Example: A Total Eclipse of the Sun

The diameter of the sun is about 400 times greater than that of the moon. By coincidence, the sun is also about 400 times farther from the earth than is the moon.
For an observer on the earth, compare the angle subtended by the moon to the angle subtended by the sun and explain why this result leads to a total solar eclipse.



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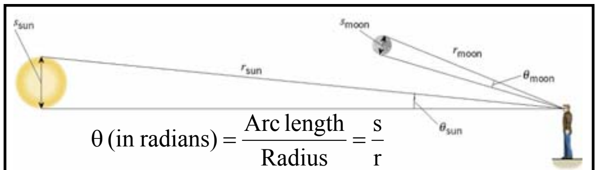

$$\theta\,(\text{in radians}) = \frac{\text{Arc length}}{\text{Radius}} = \frac{s}{r}$$

$$\theta_{sun} = \frac{s_{sun}}{r_{sun}} = \frac{400\,s_{moon}}{400\,r_{moon}} = \frac{s_{moon}}{r_{moon}} = \theta_{moon}$$

The angles subtended by the sun and the moon are approximately equal. Therefore it is possible to have a total solar eclipse

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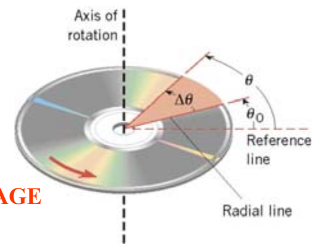
Angular Velocity and Angular Acceleration

$$\Delta\theta = \theta - \theta_o$$

How do we describe the rate at which the angular displacement is changing?



DEFINITION OF AVERAGE ANGULAR VELOCITY

$$\text{Average angular velocity} = \frac{\text{Angular displacement}}{\text{Elapsed time}}$$

$$\boxed{\overline{\omega} = \frac{\theta - \theta_o}{t - t_o} = \frac{\Delta\theta}{\Delta t}}$$

***SI Unit of Angular Velocity:* radian per second (rad/s)**

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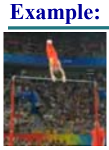
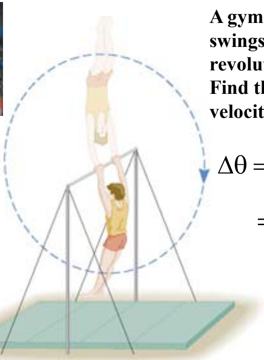
Example: Gymnast on a High Bar





A gymnast on a high bar swings through two revolutions in a time of 1.90 s. Find the average angular velocityof the gymnast.

$$\Delta\theta = -2.00\,\text{rev}\left(\frac{2\pi\,\text{rad}}{1\,\text{rev}}\right)$$

$$= -12.6\,\text{rad}$$

$$\overline{\omega} = \frac{-12.6\,\text{rad}}{1.90\,\text{s}}$$

$$= -6.63\,\text{rad/s}$$

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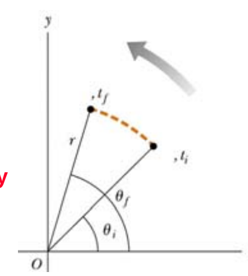
(Instantaneous) Angular Velocity

- **The average angular velocity (speed), ω, of a rotating rigid object is the ratio of the angular displacement to the time interval**

$$\overline{\omega} = \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t}$$

(instantaneous) **angular velocity**

$$\boxed{\omega = \lim_{\Delta t\to 0}\overline{\omega} = \lim_{\Delta t\to 0}\frac{\Delta\theta}{\Delta t}}$$



Radians per second is unit for ω, the angular velocity

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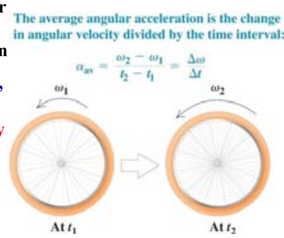
Angular Acceleration

If we follow what we studied in linear motion, when velocity changes, we call the rate of change --- acceleration

The **average angular acceleration**, α, of a rotating rigid object is the ratio of change in angular velocity to the time interval

$$\overline{\alpha} = \frac{\omega - \omega_o}{t - t_o} = \frac{\Delta\omega}{\Delta t}$$



(instantaneous)
angular acceleration

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t}$$

SI Unit of Angular acceleration: **radian per second squared (rad/s^2)**

Example: A Jet Revving its Engines

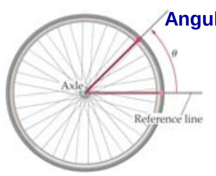
As seen from the front of the engine, the fan blades are rotating with an angular speed of -110 rad/s. As the plane takes off, the angular velocity of the blades reaches -330 rad/s in a time of 14 s. Find the angular acceleration, assuming it to be constant.

$$\overline{\alpha} = \frac{(-330\,\text{rad/s}) - (-110\,\text{rad/s})}{14\,\text{s}}$$

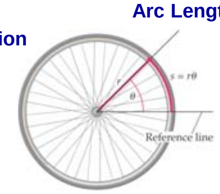
$$= -16\,\text{rad/s}^2$$

Rotational Variables

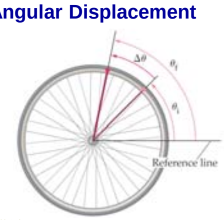
Angular Position



Arc Length



Angular Displacement



Angular speed and velocity

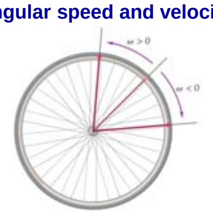
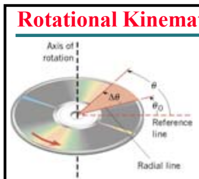


Table 8.2 Symbols Used in Rotational and Linear Kinematics

Rotational Motion	Quantity	Linear Motion
θ	Displacement	x
ω_0	Initial velocity	v_0
ω	Final velocity	v
α	Acceleration	a
t	Time	t

Rotational Kinematics and Variables



Angular Displacement θ

$$\theta\,(\text{in radians}) = \frac{\text{Arc length}}{\text{Radius}} = \frac{s}{r}$$

2 π radians = 360°

angular velocity

Unit: rad / sec

$$\omega = \lim_{\Delta t \to 0} \overline{\omega} = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$$

angular acceleration

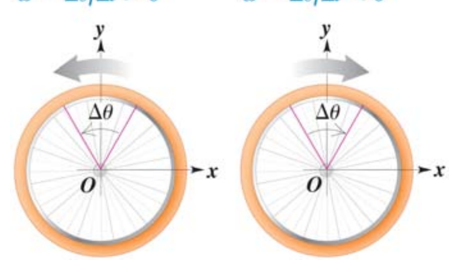
Unit: rad / sec^2

$$\alpha = \lim_{\Delta t \to 0} \overline{\omega} = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t}$$

Sign convention for angular displacement and velocity

Counterclockwise rotation positive:
$\Delta\theta > 0$, so
$\omega = \Delta\theta/\Delta t > 0$

Clockwise rotation negative:
$\Delta\theta < 0$, so
$\omega = \Delta\theta/\Delta t < 0$



Example: Rotation of a Compact Disc

A compact disc (CD) rotates at high speed while a laser reads data encoded in a spiral pattern. The disk has radius R = 6.0 cm. When data are being read, it spins at 7200 rev/min.

What is the CD's angular velocity ω in radians per second ? How much time is required for it to rotate through 90° ? If it starts from rest and reaches full speed in 4.0 s, what is its average angular acceleration?

$$\omega = \frac{\Delta\theta}{\Delta t} \qquad \omega = \frac{7200\,\text{rev}}{60\,\text{s}}\frac{2\pi\,\text{rad}}{\text{rev}} = 754\,\frac{\text{rad}}{\text{s}}$$

$$\Delta t = \frac{\Delta\theta}{\omega} = \frac{\pi/2\,\text{rad}}{754\,\text{rad/s}} = 2.1\times10^{-3}\,\text{s}$$

$$a_{av} = \frac{\Delta\omega}{\Delta t} = \frac{\omega-0}{\Delta t} = \frac{754\,\text{rad/s}}{4.0\,\text{s}} = 189\,\frac{\text{rad}}{\text{s}^2}$$

Recall the equations of 'translational' kinematics for constant acceleration

Five kinematic variables:

1. displacement, x
2. acceleration (constant), a
3. final velocity (at time t), v
4. initial velocity, v_o
5. elapsed time, t

$$a = \text{const.}$$

$$\boxed{v = v_o + at}$$

$$\boxed{x = v_o t + \tfrac{1}{2}at^2}$$

$$x = \tfrac{1}{2}(v_o + v)t \qquad v^2 = v_o^2 + 2ax$$

The Equations of Rotational Kinematics

angular acceleration: $\alpha = \text{const.}$

ANGULAR VELOCITY

ANGULAR ACCELERATION

$$\omega = \omega_o + \alpha t$$

TIME

$$\theta = \omega_o t + \tfrac{1}{2}\alpha t^2$$

ANGULAR DISPLACEMENT

$$\theta = \tfrac{1}{2}(\omega_o + \omega)t \qquad \omega^2 = \omega_o^2 + 2\alpha\theta$$

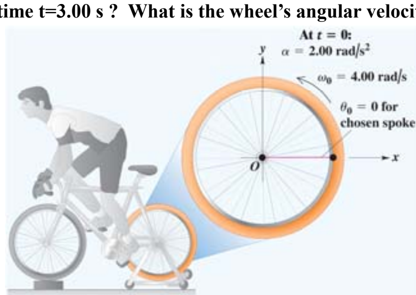
The Equations of Rotational Kinematics

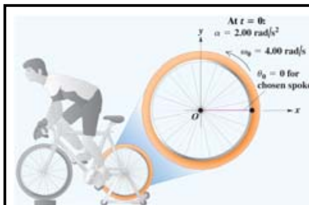
Table 8.1 The Equations of Kinematics for Rotational and Linear Motion

Rotational Motion (α = constant)		Linear Motion (a = constant)	
$\omega = \omega_0 + \alpha t$	(8.4)	$v = v_0 + at$	(2.4)
$\theta = \frac{1}{2}(\omega_0 + \omega)t$	(8.6)	$x = \frac{1}{2}(v_0 + v)t$	(2.7)
$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$	(8.7)	$x = v_0 t + \frac{1}{2}at^2$	(2.8)
$\omega^2 = \omega_0{}^2 + 2\alpha\theta$	(8.8)	$v^2 = v_0{}^2 + 2ax$	(2.9)

Example: Rotation of a Bicycle Wheel

The angular velocity of the rear wheel of a stationary exercise bike is 4.00 rad/s at time t=0, and its angular acceleration is constant and equal to 2.00 rad/s². A particular spoke coincides with the x-axis at t=0. What angle does this spoke make with the +x-axis at time t=3.00 s ? What is the wheel's angular velocity at this time ?





$$\theta_o = 0 \qquad \omega_o = 4.00\,\frac{\text{rad}}{\text{s}}$$

$$\alpha = 2.00\,\frac{\text{rad}}{\text{s}^2} \qquad t = 3.00\,\text{s}$$

$$\theta = \theta_o + \omega_o t + \frac{1}{2}\alpha t^2$$

t = 3.00 s $\Rightarrow$ $\theta = 21.0\,\text{rad}$

$$\omega = \omega_o + \alpha t$$

t = 3.00 s $\Rightarrow$ $\omega = 4.0\,\frac{rad}{s} + 2\cdot3\,\frac{rad}{s} = 10.0\,\frac{rad}{s}$

The Equations of Rotational Kinematics

Reasoning Strategy

1. Make a drawing.
2. Decide which directions are to be called positive (+) and negative (-).
3. Write down the values that are given for any of the five kinematic variables.
4. Verify that the information contains values for at least three of the five kinematic variables. Select the appropriate equation.
5. When the motion is divided into segments, remember that the final angular velocity of one segment is the initial velocity for the next.
6. Keep in mind that there may be two possible answers to a kinematics problem.

Example: Blending with a Blender

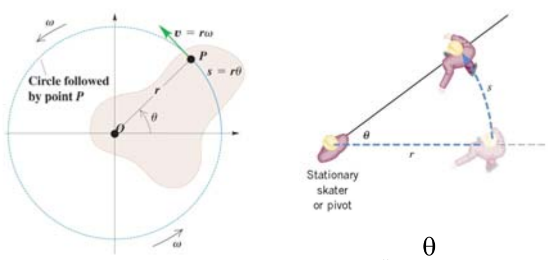
The blades are whirling with an angular velocity of +375 rad/s when the "puree" button is pushed in. When the "blend" button is pushed, the blades accelerate and reach a greater angular velocity after the blades have rotated through an angular displacement of +44.0 rad. The angular acceleration has a constant value of +1740 rad/s².
Find the final angular velocity of the blades.

$$\omega^2 = \omega_o^2 + 2\alpha\,\theta \;\Longrightarrow\; \omega = \sqrt{\omega_o^2 + 2\alpha\,\theta}$$

$$\omega = \sqrt{(375\,\text{rad/s})^2 + 2\left(1740\,\text{rad/s}^2\right)(44.0\text{rad})}$$

$$= +542\,\text{rad/s}$$

Angular Variables and Tangential Variables

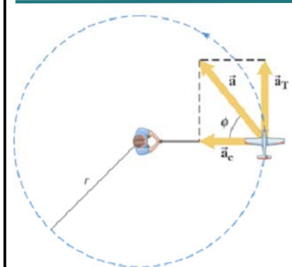


$$\omega = \frac{\theta}{t}$$

$$v_T = \frac{s}{t} = \frac{r\,\theta}{t} = r\left(\frac{\theta}{t}\right)$$

$$\boxed{v_T = r\,\omega} \quad (\omega \text{ in rad/s})$$

Centripetal Acceleration and Angular Acceleration



$$a_T = \frac{v_T - v_{To}}{t} = \frac{(r\omega)-(r\omega_o)}{t} = r\,\frac{\omega-\omega_o}{t}$$

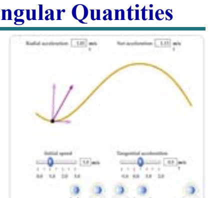
$$\alpha = \frac{\omega-\omega_o}{t}$$

$$\boxed{a_T = r\,\alpha} \quad (\alpha \text{ in rad/s}^2)$$

$$\boxed{a_c = \frac{v_T^2}{r} = \frac{(r\omega)^2}{r} = r\omega^2} \quad (\omega \text{ in rad/s})$$

Relationship between Linear and Angular Quantities

- **Displacement** $\boxed{\Delta\theta = \dfrac{\Delta s}{r}}$
- **Speed** $\boxed{\omega = \dfrac{v}{r}}$
- **Acceleration** $\boxed{\alpha = \dfrac{a_T}{r}}$ $\boxed{a_C = a_{rad} = \omega^2 r = \dfrac{v^2}{r}}$

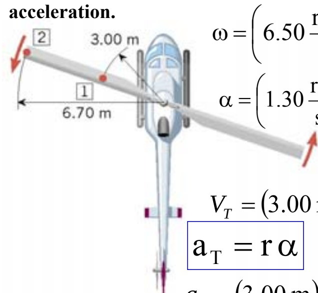


When a rigid object rotates about a fixed axis, every portion of the object has the same <u>angular speed</u> and the same <u>angular acceleration</u>

i.e. θ, ω, and α are <u>not</u> dependent upon *r*, distance form hub or axis of rotation

Example: A Helicopter Blade

A helicopter blade has an angular speed of 6.50 rev/s and an angular acceleration of 1.30 rev/s². For point 1 on the blade, find the magnitude of (a) the tangential speed and (b) the tangential acceleration.



$$\omega = \left(6.50\,\frac{\text{rev}}{\text{s}}\right)\left(\frac{2\pi\,\text{rad}}{1\,\text{rev}}\right) = 40.8\,\text{rad/s}$$

$$\alpha = \left(1.30\,\frac{\text{rev}}{\text{s}^2}\right)\left(\frac{2\pi\,\text{rad}}{1\,\text{rev}}\right) = 8.17\,\text{rad/s}^2$$

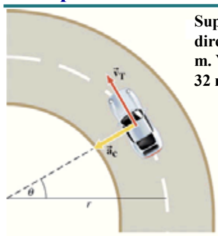
$$\boxed{v_T = r\,\omega}$$

$$V_T = (3.00\,\text{m})(40.8\,\text{rad/s}) = 122\,\text{m/s}$$

$$\boxed{a_T = r\,\alpha}$$

$$a_T = (3.00\,\text{m})\left(8.17\,\text{rad/s}^2\right) = 24.5\,\text{m/s}^2$$

Example: Acceleration of Your Car on a Circular Roadway



Suppose you are driving a car in a counterclockwise direction on a circular road whose radius is r = 390 m. You look at the speedometer and it reads a steady 32 m/s (about 72 mi/h).

a) What is the angular speed of the car?

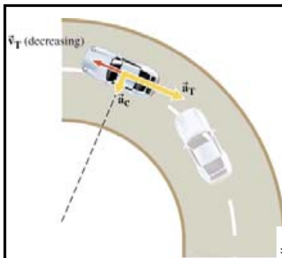
$$\omega = \frac{v_T}{r} = \frac{32\ \text{m/s}}{390\ \text{m}} = \boxed{8.2 \times 10^{-2}\ \text{rad/s}}$$

b) Determine the acceleration (magnitude and direction) of the car.

$$a_c = r\omega^2 = (390\ \text{m})\left(8.2 \times 10^{-2}\ \text{rad/s}\right)^2 = \boxed{2.6\ \text{m/s}^2}$$

centripetal

v = const. $\Rightarrow$ $a_T = 0$



c) To avoid a rear-end collision with a vehicle ahead, you apply the brakes and reduce your angular speed to 4.9 × 10-2 rad/s in a time of 4.0 s. What is the tangential acceleration (magnitude and direction) of the car?

$$\alpha = \frac{\omega - \omega_0}{t}$$

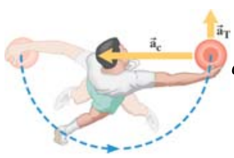
$$= \frac{4.9 \times 10^{-2}\ \text{rad/s} - 8.2 \times 10^{-2}\ \text{rad/s}}{4.0\ \text{s}}$$

$$= -8.3 \times 10^{-3}\ \text{rad/s}^2$$

$$a_T = r\alpha = (390\ \text{m})\left(8.3 \times 10^{-3}\ \text{rad/s}^2\right) = \boxed{3.2\ \text{m/s}^2}$$

Example: A Discus Thrower

Starting from rest, the thrower accelerates the discus to a final angular speed of +15.0 rad/s in a time of 0.270 s before releasing it. During the acceleration, the discus moves in a circular arc of radius 0.810 m. Find the magnitude of the total acceleration.



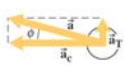
(a)

$$a_c = r\,\omega^2$$

$$a_c = (0.810\ \text{m})(15.0\ \text{rad/s})^2 = 182\ \text{m/s}^2$$

$$a_T = r\alpha = r\frac{\omega - \omega_o}{t}$$

$$a_T = (0.810\ \text{m})\left(\frac{15.0\ \text{rad/s}}{0.270\ \text{s}}\right) = 45.0\ \text{m/s}^2$$



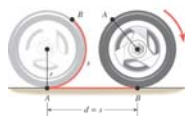
(b)

$$a = \sqrt{a_T^2 + a_c^2}$$

$$= \sqrt{(182\ \text{m/s}^2)^2 + (45.0\ \text{m/s}^2)^2} = 187\ \text{m/s}^2$$

Rolling Motion = Rotational + translational motion

A car moves with linear speed v.

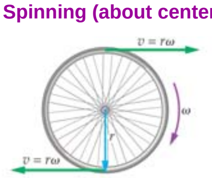


If the tires roll and do not slip, the distance d equals the circular arc length.

Rotational and Translational Velocities of a Wheel

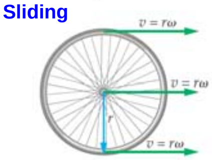
Spinning (about center)

$$\boxed{v = r\omega} \qquad \boxed{a = r\alpha}$$



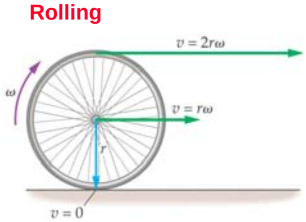
(a)

Sliding



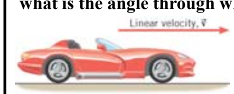
(b)

Rolling

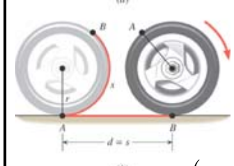


Example Rolling Motion: An Accelerating Car

An automobile, starting from rest, has a linear acceleration to the right whose magnitude is 0.800 m/s². During the next 20.0 s the tires roll and do not slip. The radius of each wheel is 0.330 m. At the end of 20.0 s, what is the angle through which each wheel has rotated?



(a)



(b)

$$\alpha = -\frac{a}{r}$$

Tire cw : $\theta < 0$

$$= -\frac{0.800\ \text{m/s}^2}{0.330\ \text{m}} = -2.42\ \text{rad/s}^2$$

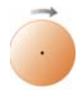
$$\theta = \omega_o t + \tfrac{1}{2}\alpha t^2 = \tfrac{1}{2}\alpha t^2$$

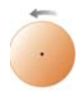
$$\theta = \tfrac{1}{2}\left(-2.42\ \text{rad/s}^2\right)\cdot(20.0\ \text{s})^2 = -484\ \text{rad}$$

Vector Nature of Angular Quantities

- Angular displacement, velocity and acceleration are all vector quantities
- Direction can be more completely defined by using the right hand rule
  - Grasp the axis of rotation with your right hand
  - Wrap your fingers in the direction of rotation
  - Your thumb points in the direction of ω

- In a, the disk rotates clockwise, the angular velocity is into the page
- In b, the disk rotates counterclockwise, the angular velocity is out of the page


(a)


(b)

Summary: Rotational versus Linear Motion

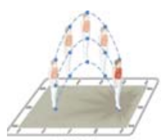
Rotational Motion About a Fixed Axis with Constant Acceleration	Linear Motion with Constant Acceleration
$\omega = \omega_i + \alpha t$	$v = v_i + at$
$\Delta\theta = \omega_i t + \frac{1}{2}\alpha t^2$	$\Delta x = v_i t + \frac{1}{2} a t^2$
$\omega^2 = \omega_i^2 + 2\alpha\Delta\theta$	$v^2 = v_i^2 + 2a\Delta x$

And for a point at a distance **r** from the rotation axis:

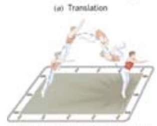
$$\Delta\theta = \frac{\Delta s}{r} \qquad \omega = \frac{v}{r} \qquad \alpha = \frac{a_T}{r}$$

Motion of Rigid Objects: Translation and Rotation

In pure translational motion, all points on an object travel on parallel paths.


(a) Translation

The most general motion is a combination of translation and rotation.


(b) Combined translation and rotation

The translational motion of the center of mass plus the rotational motion about the center of mass.