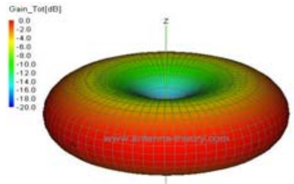
CHAPTER 11
RADIATION



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Outlines

1. **Electric Dipole radiation**
2. **Magnetic Dipole Radiation**
3. **Point Charge**
4. **Synchrotron Radiation**

What is electromagnetic radiation?

From Chap. 2 to Chap. 8, we deal with electromagnetic fields---both static and time dependent. Only in Chap. 9, we started studying electromagnetic waves that propagate through space, where $\vec{E} \perp \vec{B}$, and $\vec{S}$ carries energy away form the source to infinite far away. These waves are called electromagnetic radiation.

In chap. 9, we deal with propagation of waves through space. Here we want to study the origin of these EM radiation. We assume that sources of radiation are localized and finite.

The signature of radiation is an irreversible flow of energy away from the source.

$$P(r) = \oint \vec{S} \cdot \overrightarrow{da} = \frac{1}{\mu_o} \oint (\vec{E} \times \vec{B}) \cdot \overrightarrow{da}$$

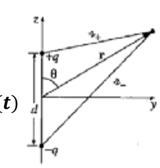
Power radiated $= \lim_{r\to\infty} P(r)$

For static fields, $E \propto \frac{1}{r^2}$ (point charge), and $B \propto \frac{1}{r^3}$ ($Dipole$) such that

$$\lim_{r\to\infty} \oint \frac{1}{r^2} \cdot \frac{1}{r^3} da \to 0$$ No radiation!!

In Jefimenko's equation, the time dependent field is $\propto \frac{1}{r}$, which is the radiation term. Here we will study several simple time-varying sources that emit radiation.

Electric dipole radiation



Assume an oscillating dipole

$q(t) = q_o cos\omega t,$ $\quad q_+ = q(t), q_- = -q(t)$

$$\vec{P}(t) = q(t)\vec{d} = P_o cos\omega t \hat{k}$$

We can write down the retarded potential as in eq. (11-5)

$$V(\vec{r}, t) = \frac{1}{4\pi\varepsilon_o}\left[\frac{q_o cos[\omega(t - {r_+}/{c})]}{r_+} - \frac{q_o cos[\omega(t - {r_-}/{c})]}{r_-}\right] \quad (1)$$

where

$$r_\pm = \sqrt{r^2 \mp r d cos\theta + \left(\frac{d}{2}\right)^2} \quad (2)$$

Law of cosines

First approximation $r \gg d$ → **Ideal dipole**

$$r_{\pm} \cong \sqrt{r^2 \mp r d \cos\theta} \cong r\left(1 \mp \frac{1}{2}\frac{d}{r}\cos\theta\right)$$ 1st order approximation

$$\Rightarrow \quad \frac{1}{r_{\pm}} \approx \frac{1}{r}\left(1 \pm \frac{1}{2}\frac{d}{r}\cos\theta\right) \quad (3)$$

$$\cos\omega\left(t - \frac{r_{\pm}}{c}\right) = \cos\omega\left(t - \frac{r}{c} \pm \frac{d}{2c}\cos\theta\right)$$

$$= \cos\left[\omega\left(t - \frac{r}{c}\right) \pm \frac{\omega d}{2c}\cos\theta\right]$$

$$= \cos\omega\left(t - \frac{r}{c}\right)\cos\left(\frac{\omega d}{2c}\cos\theta\right) \mp \sin\omega\left(t - \frac{r}{c}\right)\sin\left(\frac{\omega d}{2c}\cos\theta\right) \quad (4)$$

2nd Approximation **(Long wavelength approximation)**

$$\lambda \gg d \;\Rightarrow\; \frac{c}{\omega} \gg d \;\Rightarrow\; 1 \gg \frac{d\omega}{c}$$

$$\cos\left(\frac{\omega d}{2c}\cos\theta\right) \approx 1.0, \quad \text{and} \quad \sin\left(\frac{\omega d}{2c}\cos\theta\right) \approx \left(\frac{\omega d}{2c}\cos\theta\right)$$

Equation (4) on page 7 can be simplified

$$\cos\omega\left(t - \frac{r_{\pm}}{c}\right) = \cos\omega\left(t - \frac{r}{c}\right) \mp \left[\frac{\omega d}{2c}\cos\theta\right]\sin\omega(t - r/c)$$

So equation (1) can be re-written

$$V(\vec{r},t) = \frac{q_o}{4\pi\varepsilon_o}\left[\frac{\cos\left(t-\frac{r}{c}\right) - [\quad]\sin\omega\left(t-\frac{r}{c}\right)}{r\left(1-\frac{d}{2r}\cos\theta\right)} - \frac{\cos\omega\left(t-\frac{r}{c}\right) + [\quad]\sin\omega\left(t-\frac{r}{c}\right)}{r\left(1+\frac{d}{2r}\cos\theta\right)}\right] \quad (5)$$

where [ ] is $\left[\frac{\omega d}{2c}\cos\theta\right]$

Since

$$\frac{a}{1-x} - \frac{a}{1+x} = \frac{a(1+x) - a(1-x)}{1-x^2} = \frac{2ax}{1-x^2} \approx 2ax$$

The 1st and 3rd terms of eq. (5) can be simplified as

$$\Rightarrow \quad \frac{\cos\omega\left(t-\frac{r}{c}\right)}{r\left(1-\frac{d}{2r}\cos\theta\right)} - \frac{\cos\omega\left(t-\frac{r}{c}\right)}{r\left(1+\frac{d}{2r}\cos\theta\right)}$$

$$= 2 \cdot \frac{\cos\omega\left(t-\frac{r}{c}\right)}{r} \cdot \frac{d}{2r}\cos\theta = \frac{d \cdot \cos\theta \cdot \cos\omega\left(t-\frac{r}{c}\right)}{r^2} \quad (6)$$

and

$$\frac{a}{1-x} + \frac{a}{1+x} \approx 2a$$

The 2nd and 4th terms of equation (5) can be further simplified as

$$\frac{-[\quad]\sin\omega\left(t-\frac{r}{c}\right)}{r\left(1-\frac{d}{2r}\cos\theta\right)} + \frac{-[\quad]\sin\omega\left(t-\frac{r}{c}\right)}{r\left(1+\frac{d}{2r}\cos\theta\right)} = -2[\quad]\sin\omega\left(t-\frac{r}{c}\right) \quad (7)$$

Now, we substitute equations (6) and (7) into eq. (5)

$$V(\vec{r},t) = \frac{q_o d\cos\theta}{4\pi\varepsilon_o r}\left[-\frac{\omega}{c}\sin\omega\left(t-\frac{r}{c}\right) + \frac{1}{r}\cos\omega\left(t-\frac{r}{c}\right)\right] \quad (8)$$

This is the potential due to an oscillating dipole. We shall see later that the 1st term is the radiation term, while the 2nd term is the static dipole term. If we let $\omega \to 0$, equation (8) reduces to statics case, namely

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\frac{q_o d \cdot \cos\theta}{r^2} = \frac{1}{4\pi\varepsilon_o}\frac{\vec{P}\cdot\hat{n}}{r^2}$$

3rd Approximation $r \gg \lambda$ **(Far field)**

So equation (8) becomes

$$V(\vec{r}) = -\frac{P_o\omega}{4\pi\varepsilon_o c}\left(\frac{\cos\theta}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right] \quad (9)$$

In radiation phenomenon, there are three length scales that are important:

1. **Near field (zone)** $\quad d \ll r \ll \lambda$
2. **Intermediate field (zone)** $\quad d \ll r \approx \lambda$
3. **Radiation field (far field)** $\quad d \ll \lambda \ll r$

Vector potential

The vector potential is determined by the current

$$\vec{I}(t_r) = \frac{dq}{dt}\hat{k} = -q_o\omega\sin\omega t_r\hat{k}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_o}{4\pi}\int_{-d/2}^{d/2}\frac{-q_o\omega\sin\left[\omega\left(t-\frac{r}{c}\right)\right]}{r}dz \quad (9.1)$$

This is not an easy integral, we assume that $r \gg d$ (The ideal dipole approximation), and r will take the average value of r during the integral, we end up with

$$\vec{A}(\vec{r},t) \cong -\frac{\mu_o P_o\omega}{4\pi r}\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{k} \quad (10)$$

From equations (9) and (10), we are in a position to find the fields through the following:

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \quad \text{and} \quad \vec{B} = \nabla \times \vec{A}$$

From symmetry, we can see that both scalar and vector potentials do not have φ dependence

$$\nabla V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta}$$

$$\nabla V = -\frac{P_o \omega}{4\pi\varepsilon_o c}\left[\cos\theta\left(-\frac{1}{r^2}\sin\omega\left(t-\frac{r}{c}\right) - \frac{\omega}{rc}\cos\omega\left(t-\frac{r}{c}\right)\right)\hat{r}\right]$$

$$+\frac{P_o \omega}{4\pi\varepsilon_o c}\left[\frac{\sin\theta}{r^2}\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right]\hat{\theta} \qquad (11)$$

$\frac{1}{r} \ll \frac{c}{\omega}$

For $r \gg \lambda$, (far field approximation) the 1st and the 3rd term drop out, we end up with

$$\nabla V \cong \frac{P_o \omega^2}{4\pi\varepsilon_o c^2}\left(\frac{\cos\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{r} \qquad (12)$$

Similarly the far field approximation for $\frac{\partial \vec{A}}{\partial t}$ is

$$\frac{\partial \vec{A}}{\partial t} = -\frac{\mu_o P_o \omega^2}{4\pi r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right](\cos\theta\hat{r} - \sin\theta\hat{\theta}) \qquad (13)$$

Combining eq. (12) and (13)

$$\vec{E} = -\frac{\mu_o P_o \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta} \qquad (14)$$

Meanwhile

$$\nabla \times \vec{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right]\hat{\varphi}$$

$$= -\frac{\mu_o P_o \omega}{4\pi r}\left\{\frac{\omega}{c}\sin\theta\cos\left[\omega\left(t-\frac{r}{c}\right)\right] + \frac{\sin\theta}{r}\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right\}\hat{\varphi} \quad (15)$$

Again, for $r \gg \lambda$, we drop the 2nd term

$$\vec{B} = -\frac{\mu_o P_o \omega}{4\pi r}\frac{\omega}{c}\sin\theta \cdot \cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\varphi} \qquad (16)$$

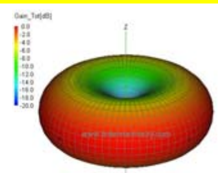
Equations (14) and (16) represent the EM waves radiate out from the oscillating electrical dipole. We can see that $\vec{E}$ and $\vec{B}$ are in phase and $\vec{E} \perp \vec{B}$.

The Poynting vector is

$$\vec{S} = \frac{1}{\mu_o}\vec{E} \times \vec{B} = \frac{\mu_o}{c}\left[\frac{P_o \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\right]^2 \hat{r} \qquad (17)$$

The intensity of the radiation is given by the time-average of the Poynting vector.

$$\langle \vec{S} \rangle = \left(\frac{\mu_o P_o^2 \omega^4}{32\pi^2 c}\right) \cdot \frac{\sin^2\theta}{r^2}\hat{r} \qquad (18)$$



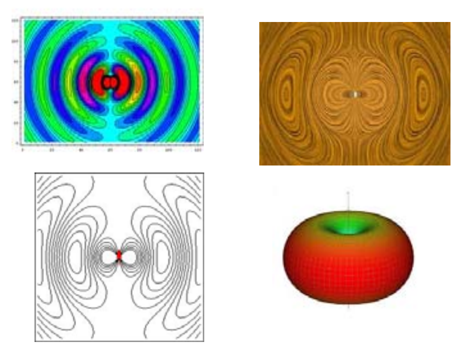
Total power radiated is given by

$$\langle P \rangle = \oint \langle \vec{S} \rangle \cdot \vec{da} = \left(\frac{\mu_o P_o^2 \omega^4}{32\pi^2 c}\right)\int \frac{\sin^2\theta}{r^2} r^2 \sin\theta d\theta d\varphi$$

$$\langle P \rangle = \frac{1}{4\pi\varepsilon_o}\frac{P_o^2 \cdot \omega^4}{3c^2} \qquad (19)$$

The electric dipole radiation has the following characteristics.

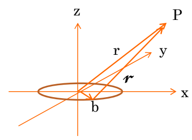
1. The radiation is along the $\hat{r}$, radially outward.
2. The $1/_{r^2}$ depend agrees with the energy conservation.
3. The $\sin^2\theta$ dependence. More energy radiated at 90°.
4. The energy radiated is $\propto \omega^4$. This is an important physics law----Rayleigh scattering. It explains why the sky is blue, and sunset is reddish.



Magnetic Dipole Radiation

A wire loop of radius b carries a current,

$$I(t) = I_o cos\omega t \quad (20)$$



The magnetic dipole associated with this current is given by

$$\vec{m}(t) = \pi b^2 \cdot I(t)\hat{k}$$

The scalar potential is zero. The vector potential is given by the retarded potential

$$\vec{A}(\vec{r},t) = \frac{\mu_o}{4\pi}\int \frac{I_o cos[\omega(t - r/c)]}{r} d\vec{\ell}' \quad (21)$$

We choose the observation point P on the x-z plane, and the direction of $\vec{A}$ at P is in the y-direction by symmetry argument

$$\vec{A}(\vec{r},t) = \frac{\mu_o I_o b}{4\pi}\hat{j}\int_0^{2\pi} \frac{cos[\omega(t - r/c)]}{r} cos\varphi' d\varphi'$$

$$r = \sqrt{r^2 + b^2 - 2rb \cdot cos\psi}$$

ψ is the angle between $\vec{r}$ and $\vec{b}$

$$\vec{r} = rsin\theta\hat{x} + rcos\theta\hat{z} \qquad \vec{b} = bcos\varphi'\hat{x} + bsin\varphi'\hat{y}$$

$$\vec{r}\cdot\vec{b} = rb \cdot cos\psi = rbsin\theta cos\varphi'$$

$$r = \sqrt{r^2 + b^2 - 2rb \cdot sin\theta \cdot cos\varphi'} \quad (22)$$

and $\qquad cos\psi = sin\theta \cdot cos\varphi'$

1st Approximation (Ideal dipole and $r \gg b$)

Equation (22) can be viewed as

$$r \approx r\left(1 - \frac{b}{r}sin\theta \cdot cos\varphi'\right)$$

$$\frac{1}{r} \approx \frac{1}{r}\left(1 + \frac{b}{r}sin\theta \cdot cos\varphi'\right)$$

$$cos\left[\omega\left(t - \frac{r}{c}\right)\right] = cos\left[\omega\left(t - \frac{r}{c}\right) + \frac{\omega b}{c}sin\theta \cdot cos\varphi'\right]$$

$$= cos[\omega(t - {}^{r}/_{c})]cos\left(\frac{\omega b}{c}sin\theta cos\varphi'\right) - sin[\omega(t - {}^{r}/_{c})]sin\left(\frac{\omega b}{c}sin\theta cos\varphi'\right)$$

See next page $\cong cos\left[\omega\left(t - \frac{r}{c}\right)\right] - sin\left[\omega\left(t - \frac{r}{c}\right)\right]\left(\frac{\omega b}{c}sin\theta cos\varphi'\right)$

2nd Approximation (Long wavelength approximation)

$\lambda \gg b$ The previous equation becomes

$$cos[\omega(t - r/c)] \approx cos\left[\omega\left(t - \frac{r}{c}\right)\right] - \frac{\omega b}{c}sin\theta cos\varphi' sin[\omega(t - r/c)]$$

Substitute the above eq. into eq. (21)

$$\vec{A}(\vec{r},t) = \frac{\mu_o m_o}{4\pi}\left(\frac{sin\theta}{r}\right)\{\frac{1}{r}cos\left[\omega\left(t - \frac{r}{c}\right)\right]$$

$$- \frac{\omega}{c}sin\left[\omega\left(t - \frac{r}{c}\right)\right]\}\hat{\varphi} \quad (23)$$

Eq. (23) should be compared with eq. (8), we can see that the 1st term is the static term while the 2nd term is the radiation term. As $\omega \to 0$, only the static term left.

3rd approximation (Far field such that $r \gg \lambda$)

We drop the 1st term of eq. (23)

$$\vec{A}(\vec{r},t) = -\frac{\mu_o m_o \omega}{4\pi c}\left(\frac{sin\theta}{r}\right)sin[\omega(t - r/c)]\hat{\varphi}$$

From $\vec{A}$ we obtain both $\vec{E}$ and $\vec{B}$ fields as follow:

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t} = \frac{\mu_o m_o \omega^2}{4\pi c}\left(\frac{sin\theta}{r}\right)cos\left[\omega\left(t - \frac{r}{c}\right)\right]\hat{\varphi} \quad (24)$$

$$\vec{B} = \nabla\times\vec{A} = -\frac{\mu_o m_o \omega^2}{4\pi c^2}\left(\frac{sin\theta}{r}\right)cos\left[\omega\left(t - \frac{r}{c}\right)\right]\hat{\theta} \quad (25)$$

The Poynting vector is defend

$$\vec{S} = \frac{1}{\mu_o}(\vec{E}\times\vec{B}) = \frac{\mu_o}{c}\left[\frac{m_o\omega^2}{4\pi c}\left(\frac{sin\theta}{r}\right)cos[\omega(t - r/c)]\right]^2 \quad (24)$$

The time-average power radiated is

$$\langle\vec{S}\rangle = \left(\frac{\mu_o m_o^2\omega^4}{32\pi^2 c^3}\right)\frac{sin^2\theta}{r^2}\hat{r} \quad (25)$$

Integrating over the solid angle, the total power radiated is

$$\langle P\rangle = \int\langle\vec{S}\rangle\cdot\vec{da} = \frac{\mu_o m_o^2\omega^4}{32\pi^2 c^3}\int\frac{sin^2\theta}{r^2}r^2 sin\theta d\theta d\varphi$$

$$\langle P\rangle = \frac{\mu_o m_o^2\omega^4}{12\pi c^3} \quad (26)$$

Electric dipole radiation vs. Magnetic dipole radiation.

$$\langle P_e \rangle = \frac{1}{4\pi\varepsilon_o}\frac{P_o^2 \cdot \omega^4}{3c^2} \quad (19) \qquad \langle P_m \rangle = \frac{\mu_o m_o^2 \omega^4}{12\pi c^3} \quad (26)$$

$$\frac{P_m}{P_e} = \left(\frac{m_o}{P_o c}\right)^2 = \left(\frac{\pi b^2 I_o}{q_o d c}\right)^2$$

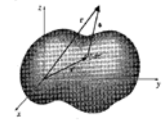
Let

$$I_o = q_o \omega \ \text{ and } \ d = \pi b$$

$$\Rightarrow \frac{P_m}{P_e} = \left(\frac{\omega b}{c}\right)^2 = \left(\frac{b}{\lambda}\right)^2 \ll 1$$

Radiation from an arbitrary source

Now we want to extend the previous derivation to an arbitrary charge distribution,



$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_o}\int \frac{\rho(\vec{r'}, t - \frac{\mathscr{r}}{c})}{\mathscr{r}} d\tau' \quad (27)$$

Where $\mathscr{r}$ is given by

$$\mathscr{r} = \sqrt{r^2 + r'^2 - 2\vec{r}\cdot\vec{r'}} \quad (28)$$

Next we will do the multiple expansion to the retarded potential of an arbitrary source—eq. (27), and apply those approximations along the way.

1st Approximation $(r \gg r')$ (Source is localized and finite)

Eq. (28) $\Rightarrow \mathscr{r} \cong r\left(1 - \frac{\vec{r}\cdot\vec{r'}}{r^2}\right)$

$$\Rightarrow \frac{1}{\mathscr{r}} \cong \frac{1}{r}\left(1 + \frac{\vec{r}\cdot\vec{r'}}{r^2}\right) \quad (29)$$

$$\rho(\vec{r'}, t - \frac{\mathscr{r}}{c}) \cong \rho\left(\vec{r'}, t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{r'}}{c}\right)$$

Define the retarded time at the origin as t_o

$$t_o \equiv t - \frac{r}{c}$$

And expand $\rho(t)$ as a Taylor series about t_o

(Taylor expansion)

$$\rho\left(\vec{r'}, t - \frac{\mathscr{r}}{c}\right) \cong \rho(\vec{r'}, t_o) + \dot{\rho}(\vec{r'}, t_o)\left(\frac{\hat{r}\cdot\vec{r'}}{c}\right) + .. \quad (30)$$

We drop all the higher order terms which amount to the long wavelength approximation $(\lambda \gg r')$.

2nd approximation

$$\dot{\rho}\left(\frac{r'}{c}\right) \gg \ddot{\rho}\left(\frac{r'}{c}\right)^2 \Rightarrow \frac{c}{|\ddot{\rho}/\dot{\rho}|} \gg r'$$

Substitute (29) (30) into eq. (27)

$$V(\vec{r},t) \cong \frac{1}{4\pi\varepsilon_o r}\{\int \rho(\vec{r'}, t_o) d\tau' + \frac{\hat{r}}{r}\cdot\int \vec{r'}\rho(\vec{r'}, t_o) d\tau' + \frac{\hat{r}}{c}\cdot\frac{d}{dt}\int \vec{r'}\rho(\vec{r'}, t_o) d\tau'\} + \cdots \quad (31)$$

Compare with equation (3.96) of Griffiths (Multipole expansion!!!)

We can see from eq. (31), the first integral is the total net charge, the second integral is the net electric dipole moment evaluated at t_o, and the 3rd term is the time derivative of the dipole moment…

$$V(\vec{r},t) \cong \frac{1}{4\pi\varepsilon_o}\left[\frac{Q}{r} + \frac{\hat{r}\cdot\vec{p}(t_o)}{r^2} + \frac{\hat{r}\cdot\dot{p}(t_o)}{rc}\right] \quad (32)$$

Next, we will try to find the vector potential $\vec{A}(\vec{r},t)$, we will start out with the retarded vector potential

$$\vec{A}(\vec{r},t) = \frac{\mu_o}{4\pi}\int \frac{\vec{J}\left(\vec{r'}, t - \mathscr{r}/c\right)}{\mathscr{r}} d\tau' \quad (33)$$

Again, this is a difficult integration. We will use similar approximation used in the dipole radiation.

The 1st approximation we made is that $r \gg r'$, so we let $\mathscr{r} \cong r$ [note that this is zero order approximation], and eq. (33) becomes

$$\vec{A}(\vec{r},t) \cong \frac{\mu_o}{4\pi r}\int \vec{J}(\vec{r'}, t_o) d\tau'$$

According to Problem 5.7 (page 223), the above eq. can be reduced to

$$\vec{A}(\vec{r},t) \cong \frac{\mu_o}{4\pi}\frac{\dot{p}(t_o)}{r} \quad (34)$$

We can see from eq.(34) that the $\dot{p}$ is already 1st order in r', so we can use the zero order approximation above.

From eq. (32) and (34), we can find $\vec{E}$ field and $\vec{B}$ field by

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}, \qquad \vec{B} = \nabla \times \vec{A}$$

3rd approximation Drop $^1/_{r^2}$ terms in $\vec{E}$ and $\vec{B}$

So we only deal with the 3rd term in eq. (32)

$$\nabla V \cong \nabla\left[\frac{1}{4\pi\epsilon_o} \cdot \frac{\hat{r}\cdot\dot{p}(t_o)}{rc}\right] \cong \frac{1}{4\pi\epsilon_o}\left[\frac{\hat{r}\cdot\ddot{p}(t_o)}{rc}\right]\left(-\frac{\hat{r}}{c}\right)$$

Combine with the results we obtained from $\frac{\partial \vec{A}}{\partial t}$ and $\nabla \times \vec{A}$, we obtain

$$\vec{E}(\vec{r},t) \cong \frac{\mu_o}{4\pi r}[\hat{r}\times(\hat{r}\times\ddot{p})] \quad (35)$$

$$\vec{B}(\vec{r},t) \cong -\frac{\mu_o}{4\pi r c}[\hat{r}\times\ddot{p}] \quad (36)$$

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Assuming that $\ddot{p}$ is in the $\hat{z}$ direction, we can write eq. (35) and (36) in the spherical coordinates

$$\vec{E}(r,\theta,t) \cong \frac{\mu_o \ddot{p}(t_o)}{4\pi}\left(\frac{\sin\theta}{r}\right)\hat{\theta} \quad (37)$$

$$\vec{B}(r,\theta,t) \cong \frac{\mu_o \ddot{p}(t_o)}{4\pi c}\left(\frac{\sin\theta}{r}\right)\hat{\varphi} \quad (38)$$

The Poynting vector follows and has the same geometry as the eq. (18) on page 16.

$$\langle\vec{S}\rangle \cong \frac{\mu_o}{16\pi^2 c}[\ddot{p}(t_o)]^2\left(\frac{\sin\theta}{r}\right)^2 \hat{r} \quad (39)$$

$$P \cong \int \vec{S}\cdot\overrightarrow{da} = \frac{\mu_o \ddot{p}^2}{6\pi c} \quad (40)$$

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Example 11.2

Assume we have a simple oscillating electric dipole

$$p(t) = p_o\cos(\omega t), \qquad \ddot{p}(t) = -\omega^2 p_o \cos(\omega t)$$

Then eq. (40) becomes

$$P_{rad}(t) \cong \frac{\mu_o}{6\pi c}\omega^4 p_o^2 \cos^2(\omega t)$$

Take time-average, we recover eq. (19) on page 25.

For point charge, we let $\vec{p}(t) = q\vec{d}(t)$, then $\ddot{p}(t) = q\vec{a}(t)$

Then eq. (40) becomes

$$P = \frac{\mu_o q^2 a^2}{6\pi c}$$

This is the famous Larmor formula: see page 38.

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Summary

1. The dominating term in the multiple expansion of arbitrary source is the electric dipole radiation. This situation is similar to the multiple expansion in the static case.
2. No monopole radiation.
3. The next higher order terms are magnetic dipole radiation and electric quadrupole radiation.
4. The Poynting vector is proportional to the square of $\ddot{p}(t)$, which is proportional to the acceleration.
5. If we let $\ddot{p}(t) = qa(t)$, eq. (37) becomes the famous Larmor formula (11.61) on page 481.

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Power radiated by a point charge

From Chap. 10 we already shown that for a point charge with an given trajectory w(t), the fields are given by (10.65) and (10.66).

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\varepsilon_o}\frac{\boldsymbol{r}}{(\vec{\boldsymbol{r}}\cdot\vec{u})^3}[\vec{u}(c^2-v^2)+\vec{\boldsymbol{r}}\times(\vec{u}\times\vec{a})]_{ret} \quad (41)$$

$$\vec{B}(\vec{r},t) = \frac{1}{c}[\hat{\boldsymbol{r}}\times\vec{E}]_{ret} \quad (42)$$

The Poynting vector is

$$\vec{S} = \frac{1}{\mu_o}(\vec{E}\times\vec{B}) = \frac{1}{\mu_o c}[\vec{E}\times(\hat{\boldsymbol{r}}\times\vec{E})]$$

Use vector triple product rule, eq. (1.17)

$$\Rightarrow \quad \vec{S} = \frac{1}{\mu_o c}[E^2\hat{\boldsymbol{r}} - \cancel{(\hat{\boldsymbol{r}}\cdot\vec{E})}\vec{E}] \quad (43)$$

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Not all terms in eq. (43) are radiation terms, as we can see from eq. (41)– only the 2nd term contains acceleration, here we will only concentrate on this term

$$\vec{E}_{rad} = \frac{q}{4\pi\varepsilon_o}\frac{\boldsymbol{r}}{(\vec{\boldsymbol{r}}\cdot\vec{u})^3}[\vec{\boldsymbol{r}}\times(\vec{u}\times\vec{a})] \quad (44)$$

The 2nd term in eq. (43) can also be dropped because radiation can not be in the $\vec{E}$ direction.

$$\vec{S}_{Rad} = \frac{1}{\mu_o c}E_{rad}^2\hat{\boldsymbol{r}} \quad (45)$$

Assume that $c \gg v$ such that

$$\vec{u} = c\hat{\boldsymbol{r}} - \vec{v} \cong c\hat{\boldsymbol{r}} \quad (46)$$

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$$\vec{E}_{rad} \cong \frac{q}{4\pi\varepsilon_o c^2 r}[\hat{r} \times (\hat{r} \times \vec{a})] = \frac{\mu_o q}{4\pi r}[(\hat{r}\cdot\vec{a})\hat{r} - \vec{a}] \quad (47)$$

Substitute (47) into (45), we obtain

$$\vec{S}_{rad} \cong \frac{1}{\mu_0 c}\left(\frac{\mu_o q}{4\pi r}\right)^2 [a^2 - (\hat{r}\cdot\vec{a})^2]\hat{r} = \frac{\mu_o q^2 a^2}{16\pi^2 c}\left(\frac{sin^2\theta}{r^2}\right)\hat{r} \quad (48)$$

It is not surprising that we obtain a radiation pattern that is quite similar to the electric dipole formula of eq. (18) on page 15.

The donut shape radiation pattern comes from the $sin^2\theta$ dependence. The angle θ is the angle between $\hat{r}$ and $\vec{a}$. Here we do not have the ω^4 dependence, because it is not a periodic motion any more.

The total power radiated is

$$P = \oint \vec{S}_{rad}\cdot d\vec{a} = \frac{\mu_o q^2 a^2}{6\pi c} \quad (49)$$

Again, this is the Larmor formula we obtained in eq. (40), page 32.

The above formula can be extended to arbitrary velocity, the result is given below

$$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_o}\frac{|\hat{r}\times(\vec{u}\times\vec{a})|^2}{(\hat{r}\cdot\vec{u})^5} \quad (50)$$

$$P = \frac{\mu_o q^2\gamma^6}{6\pi c}\left(a^2 - \left|\frac{\vec{v}\times\vec{a}}{c}\right|^2\right) \quad (51)$$

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Example 11.3

Suppose a point charge in a linear accelerator that moves in a straight line with $\vec{a}$ and $\vec{v}$ in the same direction. Find the angular distribution of the radiation and the total power emitted.

Since $\vec{a}$ and $\vec{v}$ are in the same direction, (assume c >> v)

$$\vec{u}\times\vec{a} = (c\hat{r} - \vec{v})\times\vec{a} = c(\hat{r}\times\vec{a})$$

Substitute into eq. (50) in previous page

$$\frac{dP}{d\Omega} = \frac{q^2c^2}{16\pi^2\varepsilon_o}\frac{|\hat{r}\times(\hat{r}\times\vec{a})|^2}{(c-\hat{r}\cdot\vec{v})^5} \quad (52)$$

Follow eq. (44) on page 35,

$$|\hat{r}\times(\hat{r}\times\vec{a})|^2 = a^2 - (\hat{r}\cdot\vec{a})^2$$

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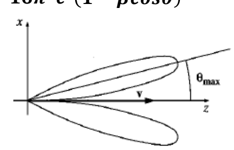
Let $\vec{v}$ and $\vec{a}$ both in the z-direction, (linear accelerator)

$$a^2 - (\hat{r}\cdot\vec{a})^2 = a^2(1-cos^2\theta) = a^2 sin^2\theta$$

$$(c - \hat{r}\cdot\vec{v}) = c(1-\beta cos\theta)$$

Substitute into eq. (50)

$$\frac{dP}{d\Omega} = \frac{\mu_o q^2 a^2}{16\pi^2 c}\frac{sin^2\theta}{(1-\beta cos\theta)^5} \quad (53)$$



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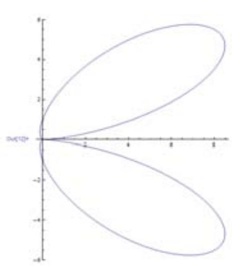
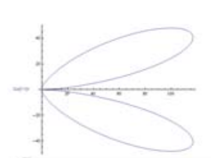
$$\frac{sin^2\theta}{(1-\beta cos\theta)^5}$$



$\beta = 0.5$



$\beta = 0.8$

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The total power radiated is given by

$$P = \int \frac{dP}{d\Omega}d\Omega = \frac{\mu_o q^2 a^2}{16\pi^2 c}\int \frac{sin^2\theta}{(1-\beta cos\theta)^5} sin\theta d\theta d\varphi \quad (54)$$

The φ integral is 2π, let cosθ=x

$$P = \frac{\mu_o q^2 a^2}{8\pi c}\int_{-1}^{1}\frac{(1-x^2)}{(1-\beta x)^5}dx \quad (55)$$

From Gradshteyn & Ryzhik Table of Integrals, let z = a+bx

$$\int \frac{1}{z^5}dx = -\frac{1}{4bz^4}$$

$$\int \frac{x^2}{z^5}dx = -\left[\frac{x^2}{2b} + \frac{ax}{3b^2} + \frac{a^2}{12b^3}\right]\frac{1}{z^4}$$

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Substitute back into equation (52) we obtain

$$P = \frac{\mu_o q^2 a^2 \gamma^6}{6\pi c} \quad (56)$$

where

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Or we can do integration by part for the integral in eq. (52)

$$\Rightarrow \int \frac{(1-x^2)}{(1-\beta x)^5}dx = \frac{4}{3}\frac{1}{(1-\beta^2)^3} \quad (57)$$

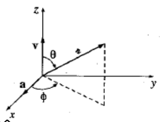
And substitute back into (55) to obtain the result of (56).

Problem 11.16

We assume that a point charge is doing a circular motion in the x-y plane about the origin. Find the radiation pattern.

We choose the axes such that

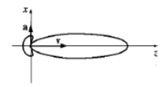
$$\vec{v} = v\hat{z}, \quad \text{and} \quad \vec{a} = a\hat{x}$$



$$\hat{r} = sin\theta \cdot cos\varphi\hat{x} + sin\theta \cdot sin\varphi\hat{y} + cos\theta\hat{z}$$

The radiation pattern is given by eq. (11.72) on page 485.

$$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_o}\frac{|\hat{r}\times(\vec{u}\times\vec{a})|^2}{(\hat{r}\cdot\vec{u})^5} \quad (58)$$



$$\vec{u} = c\hat{r} - \vec{v} = c\hat{r} - v\hat{z}$$

$$\hat{r}\cdot\vec{u} = c - vcos\theta = c(1-\beta cos\theta) \quad (59)$$

$$\hat{r}\times(\vec{u}\times\vec{a}) = (\hat{r}\cdot\vec{a})\vec{u} - (\hat{r}\cdot\vec{u})\vec{a}$$

Square the above equation, we end up with

$$|\hat{r}\times(\vec{u}\times\vec{a})|^2 = (\hat{r}\cdot\vec{a})^2u^2 - 2(\hat{r}\cdot\vec{a})(\hat{r}\cdot\vec{u})(\vec{u}\cdot\vec{a}) + (\hat{r}\cdot\vec{u})^2a^2 \quad (60)$$

$$\hat{r}\cdot\vec{a} = asin\theta cos\varphi \quad (61)$$

$$\vec{u}\cdot\vec{a} = (c\hat{r} - v\hat{z})\cdot a\hat{x} = casin\theta cos\varphi \quad (62)$$

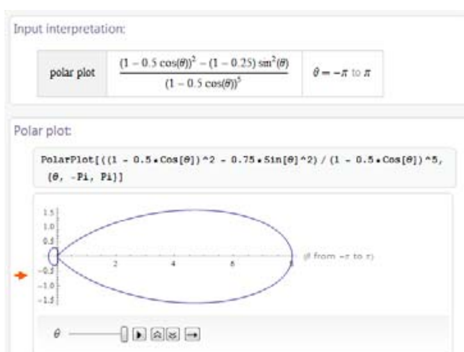
$$u^2 = c^2 - 2cvcos\theta + v^2 \quad (63)$$

Substitute (59), (61) (62) and (63) into eq. (60)

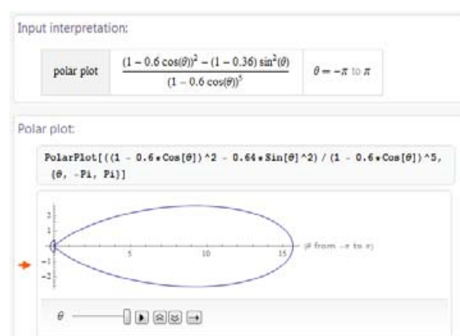
$$|\hat{r}\times(\vec{u}\times\vec{a})|^2 = a^2sin^2\theta cos^2\varphi\cdot(c^2 - 2cvcos\theta + v^2)$$
$$-2(asin\theta cos\varphi)c(1-\beta cos\theta)(casin\theta cos\varphi)$$
$$+c^2(1-\beta cos\theta)^2\cdot a^2$$
$$= a^2c^2\left[(1-\beta cos\theta)^2 - (1-\beta^2)(sin\theta cos\varphi)^2\right] \quad (64)$$

Substitute (59) and (64) back to (58), we end up with

$$\frac{dP}{d\Omega} = \frac{\mu_o q^2 a^2}{16\pi^2 c}\frac{\left|(1-\beta cos\theta)^2 - (1-\beta^2)sin^2\theta cos^2\varphi\right|}{(1-\beta cos\theta)^5}$$



$\beta = 0.5$



$\beta = 0.6$

Radiation Reaction and the Abraham-Lorentz formula

As a charged particle oscillates and radiates energy away, the particle's kinetic energy decreases. We can say the radiation exerts a force back on the particle, just like action-reaction force pair. This is called radiation reaction, we can also think of it as a damping force, which reduces the energy of the particle.

From eq. (49) on page 37, the Larmor formula gives us the power radiated by an oscillating dipole moment,

$$p = \frac{\mu_o q^2 a^2}{6\pi c}$$

This is coming from the damping term

$$\vec{F}_{rad} \cdot \vec{v} = -\frac{\mu_o q^2 a^2}{6\pi c}$$

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Assume the time average of the previous equation is approximately correct,

$$\int_{t_1}^{t_2} \vec{F}_{rad} \cdot \vec{v} dt = -\frac{\mu_o q^2}{6\pi c} \int_{t_1}^{t_2} a^2 dt \qquad (65)$$

Integration by part

$$\int_{t_1}^{t_2} a^2 dt = \int_{t_1}^{t_2} \left(\frac{d\vec{v}}{dt}\right) \cdot \left(\frac{d\vec{v}}{dt}\right) dt = \left(\vec{v} \cdot \frac{d\vec{v}}{dt}\right) - \int_{t_1}^{t_2} \frac{d^2\vec{v}}{dt^2} \cdot \vec{v} dt$$

The first term on the right side cancels out, so eq. (65) becomes

$$\int_{t_1}^{t_2} \left(\vec{F}_{rad} - \frac{\mu_o q^2}{6\pi c} \dot{a}\right) \cdot \vec{v} dt = 0$$

$$\vec{F}_{rad} = \frac{\mu_o q^2}{6\pi c} \dot{a}$$ **The Abraham-Lorentz Formula**

This is not the whole story.

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