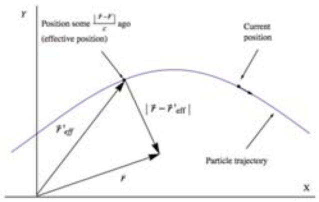
CHAPTER 10
POTENTIALS AND FIELDS



Outlines

The potential Formulism

In electrostatics, it is more convenient to use potential V, rather that the electric field $\vec{E}$

where $$\vec{E} = -\nabla V$$

And the $\vec{E}$ field satisfy

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_o} \quad \& \quad \nabla \times \vec{E} = 0$$

In magnetostatics, we have

$$\nabla \cdot \vec{B} = 0 \quad \& \quad \nabla \times \vec{B} = \mu_o \vec{J}$$

From the fact that $\nabla \cdot \vec{B} = 0$, we can define a vector potential $\vec{A}$, but we don't use $\vec{A}$ that much, because $\vec{A}$ is a vector.

And we can see in Chapter 9, we deal with fields exclusively. There, we started with Maxwell equations and then end up with Wave equations for $\vec{E}$ field and for $\vec{B}$ field.

Now we want to ask a general question regarding the presentation of materials in electrodynamics, "Is there any advantages to use potentials instead of the fields?"

The answer is YES.

Here we will try to use potentials to describe the Maxwell equations. We start with Faraday's law of induction:

Since $\nabla \cdot \vec{B} = 0 \Rightarrow \vec{B} = \nabla \times \vec{A}$

The Faraday's law can be written as:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$$

So we can define a potential such that,

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla V \quad (1)$$

Take divergence of Eq. (1) and let $\nabla \cdot \vec{E} = \rho/\varepsilon_o$, we end up with

$$-\nabla^2 V - \frac{\partial}{\partial t}(\nabla \cdot \vec{A}) = \frac{\rho}{\varepsilon_o} \quad (2)$$

The Ampere's law on the other hand can be re-written in terms of the scalar and vector potentials as follow:

$$\nabla \times \vec{B} = \mu_o \vec{J} + \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

$$\Rightarrow \nabla \times (\nabla \times \vec{A}) = \mu_o \vec{J} - \mu_o \varepsilon_o \frac{\partial}{\partial t}\left(\nabla V + \frac{\partial \vec{A}}{\partial t}\right)$$

$$\Rightarrow \left(\nabla^2 \vec{A} - \mu_o \varepsilon_o \frac{\partial^2 \vec{A}}{\partial t^2}\right) - \nabla\left(\nabla \cdot \vec{A} + \mu_o \varepsilon_o \frac{\partial V}{\partial t}\right) = -\mu_o \vec{J} \quad (3)$$

Equations (2) and (3) contain all the information of the four Maxwell equations. However, they look quite messy and may not be easy to solve. No advantages.

However in special conditions, (2)and (3) can be simplified.

Example 10.1

Given potentials

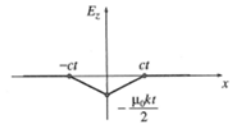
$$V = 0 \quad \& \quad \vec{A} = \begin{cases} \frac{\mu_o k}{4c}(ct - |x|)^2 \hat{z} & |x| < ct \\ 0 & |x| > ct \end{cases}$$

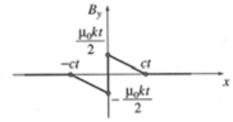
Find the source distributions, $\rho(r,t), \vec{J}(r,t), \ \& \ \vec{K}(r,t)$

The fields associated with the potentials are

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} = -\frac{\mu_o k}{2}(ct - |x|)\hat{z} \qquad (4)$$

$$\vec{B} = \nabla \times \vec{A} = \mp \frac{\mu_o k}{2c}(ct - |x|)\hat{y} \qquad (5)$$





We can show that eqs. (4) and (5) satisfy the Maxwell equations.

$$\nabla \cdot \vec{E} = 0 \quad \& \quad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = \mp \frac{\mu_o k}{2}\hat{y} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = -\frac{\mu_o k}{2c}\hat{z}, \quad \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t} = -\frac{1}{c^2}\frac{\mu_o kc}{2}\hat{z}$$

And

$$\rho(r,t) = 0 \quad \& \quad \vec{J}(r,t) = 0$$

As we can see from page 8, the magnetic field $\vec{B}$ has a discontinuity at x=0, this leads to a surface current at x=0.

$$\frac{1}{\mu_1}B_1^{\parallel} - \frac{1}{\mu_2}B_2^{\parallel} = \vec{K} \times \hat{n}$$

$$\frac{1}{\mu_o}[\mu_o kt]\hat{y} = \vec{K} \times \hat{x}$$

$$\vec{K} = kt\hat{z}$$

Gauge Transformation (Choice of Gauge)

Since $\vec{E}$ and $\vec{B}$ field are physical observables, they are uniquely defined, once the source terms and boundary conditions are fixed. For a set of field, there exist many sets of potentials that will yield the same field. We will find ways to choose a convenient potentials.

For example, in electrostatics the fields are defined through the scalar potential V and vector potential $\vec{A}$

$$\vec{E} = -\nabla V \quad \& \quad \vec{B} = \nabla \times \vec{A} \quad (6)$$

If we add a constant to V or add a gradient to $\vec{A}$

$$V' = V + C \quad \& \quad \vec{A'} = \vec{A} + \nabla\lambda \quad (7)$$

The fields remain the same.

$$\vec{E} = -\nabla V' \quad \& \quad \vec{B} = \nabla \times \vec{A'}$$

This is a remarkable result, different potentials will yield the same results for the fields. (Potential is not unique). We will try to use this concept to simplify the two equations (2) & (3) we obtained on page 5 and 6.

Now we will show that, in electrodynamics we have

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \quad \& \quad \vec{B} = \nabla \times \vec{A} \quad (8)$$

We want to choose a new set of V' and $\vec{A'}$ in such way that the fields still remain the same. We start out with the vector potential because the way we define the magnetic field does not change when we go from electrostatics to electrodynamics. (See equation (6))

We want to choose a new set of potentials, V' and $\vec{A'}$ such that the fields $\vec{E}$ and $\vec{B}$ still remain the same.

Let

$$\vec{A'} = \vec{A} + \nabla\lambda \quad (9)$$

$$V' = V - \frac{\partial \lambda}{\partial t} \quad (10)$$

Substitute (9) and (10) into equation (8)

$$\vec{E'} = -\nabla V' - \frac{\partial \vec{A'}}{\partial t} = -\nabla\left(V - \frac{\partial \lambda}{\partial t}\right) - \frac{\partial}{\partial t}(\vec{A} + \nabla\lambda)$$

$$= -\nabla V - \frac{\partial \vec{A}}{\partial t} = \vec{E}$$

Equations (9) and (10) describe a transformation of one set of potentials $(V, \vec{A})$ to another set of potentials $(V', \vec{A'})$. This is called gauge transformation.

Next, we will try to use gauge transformation to simplify equations (2) and (3).

Coulomb Gauge

In Coulomb gauge, we choose

$$\nabla \cdot \vec{A} = 0 \qquad (11) \qquad \text{Coulomb Gauge}$$

Eq. (2) becomes $\nabla^2 V = -\dfrac{\rho}{\varepsilon_o} \qquad (12)$

Eq. (3) ➡ $$\left(\nabla^2 \vec{A} - \mu_o \varepsilon_o \frac{\partial^2 \vec{A}}{\partial t^2}\right) = -\mu_o \vec{J} + \mu_o \varepsilon_o \nabla\left(\frac{\partial V}{\partial t}\right) \qquad (13)$$

We can see that in the Coulomb Gauge, the scalar potential V is very easy to solve, while the vector potential $\vec{A}$ still remains to be a difficult problem to solve.

We notice that in the Coulomb gauge, the scalar potential is still the same as in the static case:

$$V(\vec{r}, t) = \frac{1}{4\pi\varepsilon_o} \int \frac{\rho(\vec{r}', t)}{r} d\tau' \qquad (14)$$

This potential has a peculiar property, namely, the charge distribution determine the potential instantaneously. This is unusual in light of special relativity. Fortunately, the electric field contains the vector potential (see eq. (1))

The Lorentz Gauge

In the Lorentz gauge, we choose:

$$\nabla \cdot \vec{A} + \mu_o \varepsilon_o \frac{\partial V}{\partial t} = 0 \qquad (13)$$

Apply the above equation into (2) and (3), we obtain

$$\nabla^2 \vec{A} - \mu_o \varepsilon_o \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_o \vec{J} \qquad \nabla^2 V - \mu_o \varepsilon_o \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon_o}$$

Define a new operator, $\Box^2$, d'Alembertian

$$\Box^2 \equiv \nabla^2 - \mu_o \varepsilon_o \frac{\partial^2}{\partial t^2}$$

$$\Box^2 \vec{A} = -\mu_o \vec{J} \quad (14), \qquad \Box^2 V = -\frac{\rho}{\varepsilon_o} \quad (15)$$

In the 4-vector notation we define

$$A^\mu = \left(A_x, A_y, A_z, \frac{V}{c}\right) \qquad J^\mu = (J_x, J_y, J_z, c\rho)$$

$$\Box^2 A^\mu = -\mu_o J^\mu \quad (16)$$

Skip 10.1.4 Lorentz force law in potential form

Retarded Potentials

From now on we will use Lorentz Gauge only, namely

$$\nabla \cdot \vec{A} = -\frac{1}{c^2} \frac{\partial V}{\partial t}$$

So eq. (2) and (3) become

$$\nabla^2 V - \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} = -\frac{\rho(\vec{r}, t)}{\varepsilon_o} \qquad (17)$$

$$\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_o \vec{J}(\vec{r}, t) \qquad (18)$$

These equations are called inhomogeneous wave equations. They are not easy to solve. We will use a "handwaving" argument to guess a solution, and then show that the "guessed" solution is a reasonable solution.

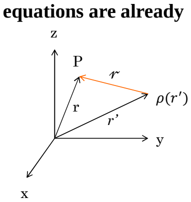
In the electrostatic case, eqs. (17) (18) reduce to Poisson equations

$$\nabla^2 V = -\frac{\rho(\vec{r})}{\varepsilon_o} \qquad \& \qquad \nabla^2 \vec{A} = -\mu_o \vec{J}$$

The solutions to the above Poisson equations are already known:

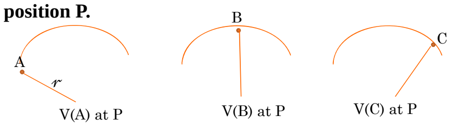
$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_o} \int \frac{\rho(\vec{r}')}{r} d\tau' \qquad (19)$$

$$\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi} \int \frac{\vec{J}(\vec{r}')}{r} d\tau' \qquad (20)$$



Now, let's start our "handwaving" argument.

First examine the following event of a charge moving along a trajectory and try to find the potential at position P.



Is it possible that $V(\vec{r},t) = \frac{1}{4\pi\varepsilon_o}\int \frac{\rho\left(\vec{r}',t\right)}{r} d\tau'$

NO! In general, this is not true! Since EM waves travel at a finite velocity. So the information of the charge will take time to reach to point P, at a time delay of $\frac{r}{c}$

Define a "retarded time"

$$t_r \equiv t - \frac{r}{c}$$

And assume the time-dependent potentials as giving by

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_o}\int \frac{\rho\left(\vec{r}',t_r\right)}{r} d\tau' \quad (21)$$

$$\vec{A}(\vec{r},t) = \frac{\mu_o}{4\pi}\int \frac{\vec{J}\left(\vec{r}',t_r\right)}{r} d\tau' \quad (22)$$

These two potentials are called retarded potentials

Next we want to prove that equations (21) and (22) indeed are solutions to the inhomogeneous wave equations (17) and (18).

Green function method

A point charge q at r', the Poisson Equation is given by

$$\nabla^2 V = q\delta(\vec{r}')$$

And the potential at $\vec{r}$, due to charge q at $\vec{r}'$ is given by

$$V_\delta(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\frac{q}{\left|\vec{r}-\vec{r}'\right|}$$

If we have a charge distribution $\rho(\vec{r}')$, the potential at $\vec{r}$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\int \frac{\rho(\vec{r}')}{\left|\vec{r}-\vec{r}'\right|} d\tau' \quad (23)$$

This is superposition theorem!!!

Now if we want to solve an arbitrary inhomogeneous differential equation, such as the following

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_o\vec{J}(\vec{r},t) \quad (24)$$

We first solve

$$\nabla^2 G - \frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} = -4\pi\delta\left(\vec{r}-\vec{r}',t-t'\right) \quad (25)$$

The solution to eq. (25) is called the Green function, and the solution to eq. (24) is given by

$$\vec{A}(\vec{r},t) = \frac{1}{4\pi}\int \left[-\mu_o\vec{J}(\vec{r}',t')\right]G(\vec{r},\vec{r}',t,t')d\tau' dt' \quad (26)$$

Now we want to show that solving eq. (17) and (18) leads to the solutions (21) and (22). We will use Green function method.

Solve $\nabla^2 G - \frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} = -4\pi\delta(\vec{r}-\vec{r}',t-t')$

The solution is given by

$$G(\vec{r},\vec{r}',t,t') = \frac{\delta(t'+\frac{r}{c}-t)}{\left|\vec{r}-\vec{r}'\right|} \quad (27)$$

Substitute eq. (27) into eq. (26)

$$\vec{A}(\vec{r},t) = \frac{1}{4\pi}\int \frac{\delta\left(t'+\frac{r}{c}-t\right)}{r}\mu_o\vec{J}\left(\vec{r}',t'\right)d\tau' dt'$$

Here we'll solve a special case, assume that $\vec{J}(r,t) = \vec{J}(r)\cdot e^{-i\omega t}$

$$\vec{A}(\vec{r},t) = \frac{\mu_o}{4\pi}\int \frac{\vec{J}(\vec{r}')}{r} d\tau' \cdot \int e^{-i\omega t'}\delta\left(t'+\frac{r}{c}-t\right)dt'$$

$$= \frac{\mu_o}{4\pi}\int \frac{\vec{J}(\vec{r}')}{r} d\tau' \cdot e^{-i\omega\left(t-\frac{r}{c}\right)} \quad \text{(Let } t_r = t-\tfrac{r}{c}\text{)}$$

$$= \frac{\mu_o}{4\pi}\int \frac{\vec{J}(\vec{r}')}{r} e^{-i\omega t_r} d\tau'$$

$$\Rightarrow \quad \vec{A}(\vec{r},t) = \frac{\mu_o}{4\pi}\int \frac{\vec{J}(\vec{r}',t_r)}{r} d\tau'$$

If the above holds, then any arbitrary function of time also will be true, because we can use Fourier series.

Next, we want to show that the retarded potentials [Eqs. (21), (22)] satisfy the inhomogeneous wave equations (17) and (18). We start with equation (21) and take gradient:

$$\nabla V(\vec{r},t) = \frac{1}{4\pi\varepsilon_o}\int\left[\frac{\nabla\rho}{\mathscr{r}} + \rho\nabla\left(\frac{1}{\mathscr{r}}\right)\right]d\tau'$$

$$\nabla\rho = \frac{\partial\rho}{\partial t_r}\left[\frac{\partial t_r}{\partial x}\hat{x} + \frac{\partial t_r}{\partial y}\hat{y} + \frac{\partial t_r}{\partial z}\hat{z}\right]$$

$$= \dot{\rho}\nabla t_r = -\frac{1}{c}\dot{\rho}[\nabla\mathscr{r}] \qquad [\,t_r \equiv t - \tfrac{\mathscr{r}}{c}\,]$$

Substitute $\nabla\mathscr{r} = \hat{\mathscr{r}}$ and $\nabla\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$

$$\nabla V = \frac{1}{4\pi\varepsilon_o}\int\left[-\frac{1}{c}\dot{\rho}\frac{\hat{\mathscr{r}}}{\mathscr{r}} - \rho\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right]d\tau' \quad (28)$$

Take divergence of equation (28), $\frac{1}{\mathscr{r}^2}$ $\quad -\frac{1}{c}\ddot{\rho}(\nabla\mathscr{r}) = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}$

$$\nabla^2 V = \frac{1}{4\pi\varepsilon_o}\int[-\frac{1}{c}\left(\dot{\rho}\left(\nabla\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}}\right)\right) - \frac{\hat{\mathscr{r}}}{\mathscr{r}}\cdot(\nabla\dot{\rho})$$

$4\pi\delta(\mathscr{r}) \qquad -\frac{1}{c}\dot{\rho}(\nabla\mathscr{r})$

$$-[\rho\left(\nabla\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) + \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot(\nabla\rho)]]d\tau' \qquad [\nabla\mathscr{r} = \hat{\mathscr{r}}]$$

Re-arrange the above expression, we end up with

$$\nabla^2 V = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} - \frac{\rho}{\varepsilon_o} \quad (29)$$

Example 10.2

An infinite straight wire carries the current

$$I(t) = \begin{cases} 0, & for\ t \le 0 \\ I_o, & for\ t > 0 \end{cases} \quad (30)$$

Assume the wire is electrically neutral, so $\rho = 0$.

$$\vec{A}(r,t) = \frac{\mu_o}{4\pi}\hat{z}\int_{-\infty}^{\infty}\frac{\vec{J}(\vec{r'},t_r)}{\mathscr{r}}d\tau' \quad (31)$$

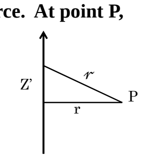
$$\vec{J}(\vec{r'},t_r) = I(t_r)\delta(x)\delta(y) \quad (32)$$

(32) Substitute into (31)

$$\vec{A}(r,t) = \frac{\mu_o}{4\pi}\int_{-\infty}^{\infty}\frac{\vec{I}(t_r)}{\mathscr{r}}dz' \quad (33)$$

Eq. (30) describes the abruptness of the turn-on of the current from the point of view of the source. At point P, the appearance of the current is given by

$$I\left(t - \frac{\mathscr{r}}{c}\right) = \begin{cases} 0, & t \le \mathscr{r}/c \\ I_o, & t > \mathscr{r}/c \end{cases}$$



So only for $\mathscr{r} < ct$, I is not zero, otherwise I = 0. Eq. (33) becomes

$$\vec{A}(r,t) = \frac{2\mu_o I_o}{4\pi}\int_0^{\sqrt{(ct)^2 - r^2}}\frac{1}{\sqrt{r^2 + z'^2}}dz'\,\hat{z}$$

$$= \frac{\mu_o I_o}{2\pi}ln\left[\frac{ct + \sqrt{(ct)^2 - r^2}}{r}\right]\hat{z} \quad (34)$$

To find the fields generated by these potentials

$$\vec{E} = -\nabla V - \frac{\partial\vec{A}}{\partial t} = -\frac{\mu_o I_o c}{2\pi\sqrt{(ct)^2 - r^2}}\hat{z} \quad (35)$$

$$\vec{B} = \nabla\times\vec{A} = \frac{\mu_o I_o}{2\pi r}\frac{ct}{\sqrt{(ct)^2 - r^2}}\hat{\varphi} \quad (36)$$

For $t \to \infty$, eq.(35) (36) approach the statics case.

$$\vec{E} = 0, \qquad \vec{B} = \frac{\mu_o I_o}{2\pi r}\hat{\varphi}$$

Do they satisfy the Maxwell equations? YES

Do they satisfy the wave equations? YES

Jefimenko's equations

When we derive the retarded potentials, Eq. (21), (22), we just used some hand-waving arguments and fortunately it works.

In general, we can not expect such simple argument would work. For example, we can not write down the field using the retarded time argument, namely

$$\vec{E}(\vec{r},t) \ne \frac{1}{4\pi\varepsilon_o}\int\frac{\rho(\vec{r'},t_r)}{\mathscr{r}^2}\hat{\mathscr{r}}\,d\tau'$$

$$\vec{B}(\vec{r},t) \ne \frac{\mu_o}{4\pi}\int\frac{\vec{J}(\vec{r'},t_r)\times\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau'$$

The exact form of the $\vec{E}$ and $\vec{B}$ field can be obtained through

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \qquad \vec{B} = \nabla \times \vec{A}$$

$$-\nabla V = \frac{1}{4\pi\varepsilon_o}\int -\nabla\left(\frac{\rho(\vec{r}', t_r)}{r}\right) d\tau'$$

$$= \frac{1}{4\pi\varepsilon_o}\int \left[\rho(\vec{r}', t_r)\left(-\nabla\frac{1}{r}\right) + \frac{1}{r}(-\nabla\rho(\vec{r}', t_r))\right]$$

$$= \frac{1}{4\pi\varepsilon_o}\int \left[\frac{\rho(\vec{r}', t_r)}{r^2}\hat{r} + \frac{\dot{\rho}(\vec{r}', t_r)}{cr}\hat{r}\right] \quad (37)$$

$$-\frac{\partial \vec{A}}{\partial t} = -\frac{\mu_o}{4\pi}\int \frac{\dot{J}'(\vec{r}', t_r)}{r} d\tau' \quad (38)$$

$$\vec{E}(\vec{r}, t) = \frac{1}{4\pi\varepsilon_o}\int \left[\frac{\rho(\vec{r}', t_r)}{r^2}\hat{r} + \frac{\dot{\rho}(\vec{r}', t_r)}{cr}\hat{r} - \frac{\dot{J}'(\vec{r}', t_r)}{c^2 r}\hat{r}\right] d\tau' \quad (39)$$

As you can see , this is a complicated equation with no utility at all. So it is not very useful. Similarly $\vec{B}$ can be found (see page 450 of Griffiths).

$$\vec{B}(\vec{r}, t) = \frac{\mu_o}{4\pi}\int \left[\frac{\vec{J}(\vec{r}', t_r)}{r^2} + \frac{\dot{J}'(\vec{r}', t_r)}{cr}\right] \times \hat{r}\, d r' \quad (40)$$

Lienard-Wiechert Potentials (Point Charge)

The Lienard-Wiechert potential is the retarded potential due to one moving point charge. In particular when v ~ c. It has its origin in the special theory of Relativity.

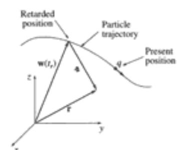
Here we will use hand-waving argument to derive it.

We started with the retarded potentials and we assume Lorentz gauge.

$$V(\vec{r}, t) = \frac{1}{4\pi\varepsilon_o}\int \frac{\rho(\vec{r}', t_r)}{r} d\tau' \quad (21)$$

$$\vec{A}(\vec{r}, t) = \frac{\mu_o}{4\pi}\int \frac{\vec{J}(\vec{r}', t_r)}{r} d\tau' \quad (22)$$

For one single point charge moving in a trajectory



$\vec{W}(t_r)$ -- position of q at time t_r

$$|\vec{r}| = |\vec{r} - \vec{w}(t_r)| = c(t - t_r)$$

$$\vec{r} = \vec{r} - \vec{W}(t_r) \quad (10.45)$$

It is important to note that at any time, there is only one point on the trajectory contribute to the potential at point P

Naively, we may think that since

$$V(\vec{r}, t) = \frac{1}{4\pi\varepsilon_o}\int \frac{\rho(\vec{r}', t_r)}{r} d\tau' = \frac{1}{4\pi\varepsilon_o}\frac{q}{r}$$

However, the above is not correct!!!

What happen is the following:

$$V(\vec{r}, t) = \frac{1}{4\pi\varepsilon_o}\frac{1}{|\vec{r} - \vec{w}(t_r)|}\int \rho(\vec{r}', t_r) d\tau' \quad (41)$$

The denominator $|\vec{r}|$ can come out of the integral without any problem, but the integral of charge is not so easy. Specifically:

$$\int \rho(\vec{r}', t_r) d\tau' \neq Q_{total}$$

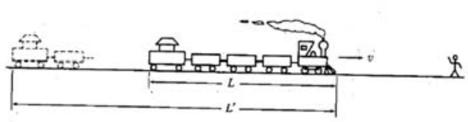
Because, t_r will be different for each different points. So we need to evaluate ρ at different times for different $\vec{r}'$ and this leads to a distortion of the total charge calculated.

For a point charge, we have

$$\int \rho(\vec{r}', t_r) d\tau' = \frac{q}{1 - \frac{\hat{r}\cdot\vec{V}}{c}} \quad (42)$$

One way to verify equation (42) is shown on page 452 of Griffiths. The time it takes the train to travel a distance $L' - L$ is the same as the time for light to travel a distance of L'.

$$\frac{L'}{c} = \frac{L' - L}{V} \rightarrow L' = \frac{L}{1 - V/c}$$



Notice that this effect does not distort the dimension perpendicular to the velocity, such that the apparent volume τ' is related to the actual volume τ by

$$\tau' = \frac{\tau}{1 - \hat{r}\cdot\vec{v}/c} \qquad (43)$$

It follows that the equations (21) (22) become

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_o}\frac{q}{r(1 - \hat{r}\cdot\vec{v}/c)} \qquad (44)$$

$$\vec{A}(\vec{r},t) = \frac{\mu_o}{4\pi}\frac{q\vec{v}}{r(1 - \hat{r}\cdot\vec{v}/c)} = \frac{\vec{v}}{c^2}V(\vec{r},t) \qquad (45)$$

These are the Lienard-Wiechert Potentials for a moving point charge.

Alternate approach to derive eqs. 44, 45

The formal way to solve equation (41) is to transform it into a new coordinate system, where time is the same such that the integration can take place.

$$\int \rho\left(\vec{r}', t_r\right) d\tau' = \int \rho'(\vec{r}_1, t_1) d\tau_1 \qquad (47)$$

$$t_r \equiv t - \frac{r}{c}$$

$$f(x) = f(a) + \frac{f'(a)}{1}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

We expand $\vec{r}_1$ in terms of $\vec{r'}$, using Taylor expansion

$$\vec{r}_1 = \vec{r}' - \vec{v}(t_r)(t_r - t_1) + (1/2)\vec{\dot{v}}(t_r)(t_r - t_1)^2 + ..$$

$$d\tau_1 \equiv \frac{\partial(x_1, y_1, z_1)}{\partial(x', y', z')} d\tau' \qquad (46)$$

where

$$\frac{\partial(x_1, y_1, z_1)}{\partial(x', y', z')} = \begin{vmatrix} \frac{\partial x_1}{\partial x'} & \frac{\partial x_1}{\partial y'} & \frac{\partial x_1}{\partial z'} \\ \frac{\partial y_1}{\partial x'} & \frac{\partial y_1}{\partial y'} & \frac{\partial y_1}{\partial z'} \\ \frac{\partial z_1}{\partial x'} & \frac{\partial z_1}{\partial y'} & \frac{\partial z_1}{\partial z'} \end{vmatrix}$$

$$\frac{\partial x_1}{\partial x'} = 1 - v'_x\frac{\partial t_r}{\partial x'} + \cdots \qquad \frac{\partial x_1}{\partial y'} = 0 - v'_x\frac{\partial t_r}{\partial y'} + \cdots$$

And $t_r = t - \frac{\left|\vec{r} - \vec{r'}\right|}{c}$

Then $\frac{\partial t_r}{\partial x'} = \frac{r_x}{c}$; $\frac{\partial t_r}{\partial y'} = \frac{r_y}{c}$; $\frac{\partial t_r}{\partial z'} = \frac{r_z}{c}$

$\hat{r} = r_x\hat{\imath} + r_y\hat{\jmath} + r_z\hat{k}$ and it is the unit vector in the $\vec{r} - \vec{r'}$ direction.

$$\frac{\partial(x_1, y_1, z_1)}{\partial(x', y', z')} = 1 - \frac{\vec{v}\cdot\hat{r}}{c} + \cdots = \frac{d\tau_1}{d\tau'}$$

$$d\tau' = \frac{d\tau_1}{1 - \hat{r}\cdot\vec{v}/c}$$

Example 10.3

Find the potentials of a point charge moving with a constant velocity.

Since it is a constant velocity motion, the trajectory is a straight line. Let's choose the line such that it pass through the origin at t = 0.

$$\vec{w}(t) = \vec{v}t$$

$$r = |\vec{r} - \vec{v}t_r| = c(t - t_r) \qquad (10.44)$$

Square above eq. and then solve for t_r (See page 49).

Since t_r is the retarded time, we choose

$$t_r = t - \frac{r}{c}$$

We first compute the retarded time, using (10.44)

$$r^2 - 2\vec{r}\cdot\vec{v}t_r + v^2t_r^2 = c^2\left(t^2 - 2tt_r + t_r^2\right) \quad (10.48)$$

$$t_r = \frac{(c^2t - \vec{r}\cdot\vec{v}) \pm \sqrt{(c^2t - \vec{r}\cdot\vec{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)}}{c^2 - v^2}$$

Next

$$r\left(1 - \frac{\hat{r}\cdot\vec{v}}{c}\right) = c(t - t_r)\left[1 - \frac{\vec{v}}{c}\cdot\frac{(\vec{r} - \vec{v}t_r)}{c(t - t_r)}\right]$$

$$= c(t - t_r) - \frac{\vec{v}\cdot\vec{r}}{c} + \frac{v^2}{c}t_r$$

$$= \frac{1}{c}\left[(c^2t - \vec{v}\cdot\vec{r}) - (c^2 - v^2)t_r\right]$$

$$= \frac{1}{c}\sqrt{(c^2t - r\cdot v)^2 + (c^2 - v^2)(r^2 - c^2t^2)} \qquad (2)$$

Substitute the above into (10.46) and (10.47), we end up with:

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_o}\frac{qc}{\sqrt{(c^2t - \vec{r}\cdot\vec{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)}}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_o}{4\pi}\frac{qc\vec{v}}{\sqrt{(c^2t - \vec{r}\cdot\vec{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)}}$$

Now if we let $\vec{v} = v\hat{\imath}$ and $\vec{r} = z\hat{k}$

$$V(z,t) = \frac{1}{4\pi\varepsilon_o}\frac{qc}{\sqrt{c^4t^2 + (c^2 - v^2)(z^2 - c^2t^2)}}$$

$$= \frac{1}{4\pi\varepsilon_o}\frac{q}{\sqrt{z^2 + \left(v^2t^2 - \frac{v^2z^2}{c^2}\right)}}$$

At t = 0

$$V(z,0) = \frac{1}{4\pi\varepsilon_o}\frac{q}{z\sqrt{1 - v^2/c^2}} = \frac{1}{4\pi\varepsilon_o}\frac{q'}{z}$$

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The fields due to Lienard-Wiechert potentials

Once the Lienard-Wiechert potentials are known, the fields can be obtained through

$$\vec{E} = -\nabla V - \frac{\partial\vec{A}}{\partial t} \quad \& \quad \vec{B} = \nabla\times\vec{A}$$

Conceptually, it is straight forward and simple. But in reality, it is an exercise in vector calculus. See page 456-457 of Griffiths for details. In the end we have

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\varepsilon_o}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\left[\vec{u}(c^2 - v^2) + \vec{\mathscr{r}}\times(\vec{u}\times\vec{a})\right]_{ret} \quad (47)$$

$$\vec{B}(\vec{r},t) = \frac{1}{c}\left[\hat{\mathscr{r}}\times\vec{E}\right]_{ret} \quad (48)$$ and $\vec{u} \equiv c\hat{\mathscr{r}} - \vec{v}$

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There are two terms in eq. (47)

Let $\vec{u} = c\hat{\mathscr{r}} - \vec{v}$

1st term

$$\frac{q}{4\pi\varepsilon_o}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\left[\vec{u}\cdot(c^2 - v^2)\right]_{ret}$$

This eq. does not depend on acceleration. It is called generalized Coulomb field, because if v=0, this part goes back to the static field.

2nd term

The 2nd term depends on acceleration. This is called radiation field.

$$\vec{E}_{radiation} \propto \frac{1}{\mathscr{r}}$$

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The static field satisfies the Gauss Law. No energy radiated out.

For the radiation field, it does not satisfy the Gauss law. But it does agree with conservation of energy.

$$\int(energy\ flux)\cdot\vec{da} = constant$$

$$[E_{rad}]^2 \propto [energy\ flux] \propto \frac{1}{\mathscr{r}^2}$$

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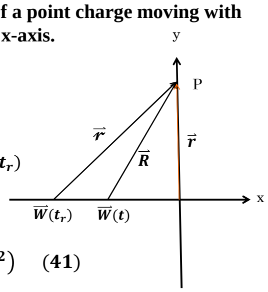
Example 10.4

Calculate the $\vec{E}$ and $\vec{B}$ field of a point charge moving with a constant velocity along the x-axis.

$$\vec{W}(t_r) = \vec{v}t_r = vt_r\hat{x}$$

$$\vec{v}(t - t_r) = \frac{\vec{v}}{c}\mathscr{r}$$

$$\vec{\mathscr{r}} = \vec{r} - \vec{W}(t_r)\ ;\ \mathscr{r} = c(t - t_r)$$



Equation (10.72) becomes

$$\vec{E} = \frac{q}{4\pi\varepsilon_o}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\vec{u}(c^2 - v^2) \quad (41)$$

$\vec{u} \equiv c\hat{\mathscr{r}} - \vec{v}$, $\quad \vec{\mathscr{r}} - \frac{\vec{v}}{c}\mathscr{r} = \frac{\mathscr{r}}{c}[c\hat{\mathscr{r}} - \vec{v}] = \frac{\mathscr{r}}{c}\vec{u} = \vec{R}$

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The direction of the non-radiative $\vec{E}$ field is in the direction of $\vec{R}$, namely from the present position of the charge. This is an extraordinary coincidence, since the "message" came from the retarded position.

$$\vec{\mathscr{r}}\cdot\vec{u} = \vec{\mathscr{r}}\cdot\frac{c}{\mathscr{r}}\vec{R} = c(\hat{\mathscr{r}}\cdot\vec{R}) = c\sqrt{R^2 - \frac{v^2}{c^2}\mathscr{r}^2\sin^2\theta'}$$

But $\quad \mathscr{r}\sin\theta' = R\sin\theta$

$$\vec{\mathscr{r}}\cdot\vec{u} = Rc\sqrt{1 - \frac{v^2}{c^2}\sin^2\theta}$$

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\varepsilon_o}\frac{\left(1 - \frac{v^2}{c^2}\right)}{\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{3/2}}\frac{\hat{R}}{R^2}$$

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New notes added

From eq. (2) of slide 42, we can see that

$$\frac{1}{c}\left[\left(c^2 t - \vec{r}\cdot\vec{v}\right) - \left(c^2 - v^2\right)t_r\right]$$

$$=\frac{1}{c}\sqrt{(c^2 t - \vec{r}\cdot\vec{v})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}$$

Re-arrange the above equation, and solve the quadratic equation of t_r, we end up with

$$t_r = \frac{\left(c^2 t - \vec{r}\cdot\vec{v}\right) \pm \sqrt{(c^2 t - \vec{r}\cdot\vec{v})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}}{c^2 - v^2}$$

To fix the sign, consider the limit, v = 0

$$t_r = \frac{c^2 t \pm \sqrt{(c^2 t)^2 + c^2(r^2 - c^2 t^2)}}{c^2} = t \pm \frac{r}{c}$$