CHAPTER 5
MAGNETOSTATICS

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A brief history of magnetostatics

The Chinese compass was invented around 4th century BC.

In 1600, William Gilbert published “De Magnete”, one of the first book on electricity and magnetism. Gilbert was regarded by some as the father of electricity and magnetism.

In 1820, Hans Oersted discovered the relationship between electric current and magnetic field, namely a current carrying wire will deflect the needle of a compass.

In the same year, Jean-Baptiste Biot and Felix Savart found that the magnetic field varies inversely with the distance from the wire and curl around the current. The equation describing the magnetic field generated by a current bears their names.

\[ \mathbf{B}(r) = \frac{\mu_0 J}{4\pi} \left( \frac{d\mathbf{l} \times \hat{r}}{r^2} \right) \]

Ampere learned about Oersted’s result in 1820 and started to develop a theory to understand the relationship of electricity and magnetism. Ampere showed that two parallel wires carrying current will attract or repel each other depending on whether the currents are in the same or opposite direction.

In 1826, Ampere discovered the Ampere’s law.

\[ \int \mathbf{B} \cdot d\mathbf{l} = \mu \mathbf{J}_{\text{enc}} \]

In 1831, Michael Faraday proposed that change in the magnetic flux can also produce electric field—Faraday’s law of induction. The “field line” concept also came from Faraday.

\[ \mathbf{E} = \frac{\mathbf{d\Phi}}{\mathbf{dt}} \]

Then in 1864, James Maxwell published his Maxwell’s equations which combined Gauss’s law, Faraday’s law, and the modified Ampere’s law. Maxwell coined the phrase “the electromagnetic field” for his theory.
Maxwell Equations

- $\oint E \cdot d\mathbf{a} = \frac{Q_{\text{in}}}{\varepsilon_0}$
- $\oint B \cdot d\mathbf{a} = 0$
- $\oint E \cdot d\mathbf{l} = \frac{d\Phi_B}{dt}$
- $\oint B \cdot d\mathbf{l} = \mu_0 J + \mu_0 \frac{d\Phi_E}{dt}$

\[ \nabla \cdot E = \frac{\rho}{\varepsilon_0} \]
\[ \nabla \cdot B = 0 \]
\[ \nabla \times E = -\frac{\partial B}{\partial t} \]
\[ \nabla \times B = \mu_0 J + \mu_0 \frac{dE}{dt} \]

Let's put things into perspective.

Electrons were discovered by J. J. Thomson in 1897. He found that the cathode rays in a cathode ray tube travel further in air than ions. He estimated that the mass of cathode rays to be at least 1000 times lighter than hydrogen.

In 1887, Michelson and Morley performed the experiment bears their name. They show that there is no “aether” and pave the way for the development of special relativity.

The Lorentz force law

When a charge q moving in a magnetic field $\mathbf{B}$, it experiences a force called the Lorentz force

\[ \mathbf{F}_m = q \cdot \mathbf{V} \times \mathbf{B} \quad (1) \]

Since the Lorentz force $\mathbf{F}_m$ is always perpendicular to the velocity, therefore no work done by Lorentz force.

If both $\mathbf{E}$ and $\mathbf{B}$ are present

\[ \mathbf{F} = q[\mathbf{E} + (\mathbf{V} \times \mathbf{B})] \]

Example 5.2 Cycloid motion

A uniform magnetic field is in the x-direction while another uniform electric field is in the z-direction as shown. Find the trajectory of a charge particle starting at rest at the origin.

Initially at rest at the origin, the charge will experience a force in the z-direction, so initial velocity will be in the z-direction. The Lorentz force will move the charge in the y-direction.

The solution to the previous equation can be

\[ y(t) = C_1 \cos \omega t + C_2 \sin \omega t + \frac{e}{\mu_0} t + C_3 \]

Substitute back into (2), we arrive at

\[ \ddot{x} = -\omega^2 (C_2 \cos \omega t - C_1 \sin \omega t) \]

Apply the initial conditions, at $t = 0, \dot{y} = 0, \text{ and } \mathbf{V} = 0$.

\[ C_1 + C_3 = 0 \quad \text{and} \quad C_2 + C_4 = 0 \]
\[ C_2 \omega + \frac{e}{\mu_0} = 0 \quad \text{and} \quad -C_1 \omega = 0 \]

\[ C_1 = C_3 = 0 \quad C_2 = -\frac{e}{\mu_0} \quad C_4 = \frac{e}{\mu_0} \]
Let $E = \frac{B}{\omega} R$

$(y - R \omega t)^2 + (z - R)^2 = R^2$

This is the equation for a circle of radius $R$ with the center at $(R, R, R)$

**Current & Current density**

Current in a wire (1D) is defined as the amount of charges passing through a point per unit time.

$I = \frac{Q}{t}$ Ampere = Coulomb per second

If line charge density is $\lambda$ and velocity is $V$,

$I = \lambda V \rightarrow \vec{I} = \lambda \vec{V}$

Since the current is almost always confined in a conducting wire, we can use the direction of the wire to indicate the direction of current

$\vec{I} = I \vec{dI}$

The concept of Lorentz force on a “charge” can be extended to “current”, since “current” can be viewed as “charge density” times velocity.

$F_{mag} = \int (\vec{V} \times \vec{B}) d\vec{l} = \int (\vec{V} \times \vec{B}) \lambda d\vec{l} = \int (\vec{I} \times \vec{B}) d\vec{l}$

Since direction of $\vec{I}$ is the same as $d\vec{l}$, and $I$ is usually a scalar constant,

$F_{mag} = \int I(\vec{dl} \times \vec{B}) = I \int (\vec{dl} \times \vec{B})$

When the loop starts to rise, the charge no longer moving horizontally, it acquires a vertical component $u$, so $F_{mag}$ also tilts left as shown. $F_{mag}$ is still perpendicular to $F$, no work done on charge $q$. The vertical component, $q u B$ which can be written as

$F_{mag} = quB = \lambda awB = I Ba$

The horizontal force, $F_h = quB = \lambda awB$, so the work done by battery to overcome $F_h$ is

$W_{battery} = \lambda ab \int uwdt = IBa h$

This is similar to sliding a block up a frictionless ramp, pushing horizontally

Example 5.3

A rectangular loop of wire, supporting a mass $m$, hanging vertically with one end in a uniform magnetic field as shown. What current would be needed to balance the gravitational force?

$F_{mag} = 1B a = mg \rightarrow I = \frac{mg}{Ba}$

What happens if we increase the current?

The magnetic force now will be larger than gravitational force $mg$, and the mass will rise.

Does that mean the magnetic force $F_{mag}$ actual do work?
When charge flows on the surface of a conductor (two dimensional), we describe it by the “surface current density” \( K \) as
\[
K = \frac{\partial i}{\partial t} \quad K = \sigma V
\]
\[
\mathbf{F}_{\text{mag}} = \int (\mathbf{K} \times \mathbf{B}) \, da
\]
When the flow of charge is 3D, we describe it by the volume current density, \( \mathbf{j} \)
\[
\mathbf{j} = \frac{1}{\mathcal{A}} \oint \mathbf{A} \cdot d\mathbf{a} = \frac{\mathbf{q}}{v} \quad \mathbf{j} = \rho \mathbf{V}
\]
\[
\mathbf{F}_{\text{mag}} = \int \rho \mathbf{V} \times \mathbf{B} \, d\tau = \int (\mathbf{j} \times \mathbf{B}) \, d\tau
\]

The Biot-Savart Law
Steady current is the source that produces magnetic field in magnetostatics. It plays exactly the same role as stationary charge in electrostatics. The Biot-Savart law plays the exactly same role in magnetostatics as the Coulomb’s law which links the source to the field.

<table>
<thead>
<tr>
<th>Electrostatics</th>
<th>Magnetostatics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Source</td>
<td>Steady current</td>
</tr>
<tr>
<td>Equation</td>
<td>Biot-Savart Law</td>
</tr>
<tr>
<td>Field*</td>
<td>Diverge from a point</td>
</tr>
</tbody>
</table>

Field*—Electric field produced by one point charge, or magnetic field produced by a current on a straight wire.

Example 5.5 Find the magnetic field a distance \( s \) from a long straight wire carrying a steady current \( I \).

The magnetic field is given by
\[
\mathbf{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{r} \times \hat{r}}{r^2}
\]
\[
|d\mathbf{r} \times \hat{r}| = d\mathbf{r} \cdot \sin \alpha = d\mathbf{r} \cos \theta
\]
\[
l' = s \cdot \tan \theta, \quad \text{and} \quad d\mathbf{r}' = s \cdot d(\tan \theta) = \frac{s \cdot d\theta}{\cos^2 \theta}
\]
\[
\frac{s}{r} = \cos \theta \quad \text{and} \quad \frac{1}{r^2} = \frac{\cos^2 \theta}{s^2}
\]

From definition of current density
\[
I = \int I d\mathbf{a}_x = \int \mathbf{j} \cdot d\mathbf{a}
\]
From divergence theorem, we have
\[
\int \mathbf{j} \cdot d\mathbf{a} = \int (\nabla \cdot \mathbf{j}) d\tau
\]
From conservation of charge, the amount of charge leaving a volume
\[
\int \mathbf{j} \cdot d\mathbf{a} = -\frac{d}{dt} \int \rho d\tau
\]
Combine with previous equation, we have
\[
\nabla \cdot \mathbf{j} = \frac{d\rho}{dt}
\]
For magnetostatics
\[
\nabla \cdot \mathbf{j} = 0
\]

The magnetic field produced by a steady current,
\[
\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \left( \mathbf{r} \times \int \frac{d\mathbf{r}' \cdot \hat{r}}{r'^2} \right)
\]
where \( \mu_0 \) is the permeability of the free space.
\[
\frac{1}{\sqrt{4\pi \mu_0}} = 2.998 \times 10^8 \text{ m/s} \quad \text{(Speed of light)}
\]
The unit of magnetic field \( \mathbf{B} \)
1 tesla = \( \frac{N}{A \cdot m} \) (MKS unit, or SI unit)
1 Gauss = \( \frac{\text{dyne}}{\text{emu} \cdot \text{cm}} \) (cgs unit)

\[
B = \frac{\mu_0 I}{4\pi s} \cos^2 \theta \cdot \frac{s}{\cos^2 \theta} \cdot d\theta \cdot \cos \theta
\]
For infinite straight line,
\[
B = \frac{\mu_0 I}{2\pi s} \frac{1}{s^2} \frac{\cos^2 \theta}{\cos^2 \theta}
\]
Now if we have two long parallel wires carrying current \( I_1 \) and \( I_2 \), the force between them is
\[
\mathbf{F} = \int l (d\mathbf{r} \times \mathbf{B})
\]
Example 5.6 Find the magnetic field above the center of a circular loop.

\[
B_x(x) = \frac{\mu_0 I}{4\pi} \int \frac{dl'}{r'^2} \cos \theta
\]

\[
B(x) = \frac{\mu_0 I}{4\pi} \left( \frac{2\pi R}{R^2 + x^2} \right) \frac{R}{r}
\]

We can show that in cylindrical coordinates

\[
\mathbf{B} = \frac{\mu_0 I}{2\pi} \hat{\phi}, \quad d\ell = dr \hat{r} + r d\phi \hat{\phi} + dx \hat{z}
\]

Substitute into the equation below

\[
\oint \mathbf{B} \cdot d\ell = \oint \frac{\mu_0 I}{2\pi} \cdot d\phi = \mu_0 I
\]

Formal derivation of \( \nabla \times \mathbf{B} \) and \( \nabla \cdot \mathbf{B} \)

We start out with the Biot-Savart Law

\[
\mathbf{B}(r') = \frac{\mu_0}{4\pi} \int \frac{j'(r') \times \hat{r}}{r'^3} \, dr'
\]

\[
\nabla \cdot \mathbf{B} = \frac{\mu_0}{4\pi} \nabla \cdot \left( \int j'(r') \times \hat{r} \right) \, dr'
\]

Apply product rule (4) on page 21 of Griffiths to eq. (1) above

\[
\nabla \cdot \left( \int j'(r') \times \hat{r} \right) = \int (\nabla \cdot \hat{r}) \, j'(r') - \int j'(r') \cdot (\nabla \times \hat{r})
\]

This is Ampere's law. This is true for any shape of closed loop around the current.

First, we can switch from \( \nabla \) to \( \nabla' \) and change a sign as below

\[
-\left( \nabla \cdot \mathbf{B} \right) = \left( \nabla' \cdot \mathbf{B} \right) + \left( \nabla' \times \mathbf{B} \right) \cdot \hat{r}
\]

Next, we will look at the x-component of the above equation and use integration by parts

\[
\nabla' \cdot \left( \frac{x-x'}{r^3} \right) = \left( \nabla' \cdot \mathbf{B} \right) \frac{x-x'}{r^3} + \left( \nabla' \times \mathbf{B} \right) \cdot \hat{r}
\]

The integrand can be re-arranged

\[
\nabla' \cdot \left( \frac{x-x'}{r^3} \right) = \left( \nabla' \cdot \mathbf{B} \right) \frac{x-x'}{r^3} \frac{x-x'}{r^3} + \frac{x-x'}{r^3} \left( \nabla' \times \mathbf{B} \right) \cdot \hat{r}
\]

The second term on the right hand side goes to zero can be seen on next page.
Applications of Ampere’s Law

The Ampere’s law can be written in integral form

\[ \oint B \cdot dl = \mu_0 I_{enc} \]

Ampere’s Law in magnetostatics plays exactly the same role as the Gauss’s Law in electrostatics. But Ampere’s law is a line integral while the Gauss’s law is a surface integral.

<table>
<thead>
<tr>
<th>Electrostatics</th>
<th>Coulomb’s Law</th>
<th>Gauss’s Law</th>
</tr>
</thead>
<tbody>
<tr>
<td>Magnetostatics</td>
<td>Biot-Savart Law</td>
<td>Ampere’s Law</td>
</tr>
</tbody>
</table>

Example 5.8 Find the magnetic field of an infinite uniform surface current \( I = \kappa \) flowing over the xy plane.

The way we solve this problem is very similar to the infinite uniform surface charge problem of Example 2.4 on page 73.

Apply Ampere’s Law, we have

\[ \oint B \cdot dl = 2BI = \mu_0 I_{enc} = \mu_0 I \]

\[ B = \begin{cases} \left( \frac{\mu_0}{2} \right) k\hat{y} & \text{for } z < 0 \\ \left( \frac{\mu_0}{2} \right) k\hat{y} & \text{for } z > 0 \end{cases} \]

Example 5.9 Find the magnetic field of a very long solenoid consisting of \( n \) closely wound turns per unit length.

(a) First we argue that there is no magnetic field in the radial direction.
(b) Second we argue that there is no magnetic field in the azimuthal direction (\( \phi \)).
(c) Third, we argue that outside the solenoid the magnetic field is equal to zero.
(d) Last we argue that the magnetic field inside the solenoid is uniform and in the z-direction.
(e) Use Ampere’s Law, we found the field inside to be

\[ B_{in} = \mu_0 nIz \]

Magnetostatics and Electrostatics

<table>
<thead>
<tr>
<th>Magnetostatics</th>
<th>Electrostatics</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \nabla \cdot B = 0 )</td>
<td>( \nabla \cdot E = \rho/\epsilon_0 )</td>
</tr>
<tr>
<td>( \nabla \times B = \mu_0 J )</td>
<td>( \nabla \times E = 0 )</td>
</tr>
<tr>
<td>( B = \nabla \times A )</td>
<td>( E = -\nabla V )</td>
</tr>
<tr>
<td>( \oint B \cdot dl = \mu_0 I_{enc} )</td>
<td>( \oint E \cdot dA = Q_{en}/\epsilon_0 )</td>
</tr>
<tr>
<td>( B = \frac{\mu_0}{4\pi} \int \frac{J \cdot d^2r}{r^2} )</td>
<td>( E = \frac{1}{4\pi\epsilon_0} \int \frac{\rho d^2r}{r^2} )</td>
</tr>
</tbody>
</table>

Re-arrange the previous equation, we can see that

\[ B = \nabla \times \left( \frac{\mu_0}{4\pi} \int \frac{J \cdot d^2r}{r} \right) \]

We let the expression in the parenthesis equal to the vector potential of the field. For comparison, electric potential, \( V(r) \) is also shown below.

\[ \vec{A} = \frac{\mu_0}{4\pi} \int \frac{\rho d^2r}{r} \]

If we add a gradient term to \( \vec{A} \), and let \( \vec{A}' = \vec{A} + \nabla \lambda \), the magnetic field \( \vec{B} \) will remain the same as shown below.

\[ B = \nabla \times \vec{A}' = \nabla \times \vec{A} + \nabla \times (\nabla \lambda) = \nabla \times \vec{A} \]
The situation here is similar to the electrostatics case, where adding a constant to the potential does not change the electric field at all.

In magnetostatics, adding a gradient term to the vector potential $\mathbf{A}'$, is called gauge transformation.

$$ \mathbf{A} = \mathbf{A}' + \nabla \lambda $$

It can be used to choose a particular vector potential. For example, we can choose a vector potential that is divergence-free.

$$ \nabla \cdot \mathbf{A} = 0 $$

This will lead to

$$ \nabla^2 \lambda = -\nabla \cdot \mathbf{A} $$

Example 5.11. A spherical shell of radius $R$, carrying a uniform surface charge $\sigma$, is set spinning at angular velocity $\omega$. Find the vector potential at point $r$.

Rotate the axis such that the $\hat{z}$ direction is the same as the $\hat{r}$. The surface current is given by $\mathbf{j} = \sigma \hat{V}$, and $\hat{V}$ is the velocity of the shell,

$$ \hat{V} = \hat{\omega} \times \hat{r} $$

The $\hat{\omega}$ is pointing at an angle $\psi$ from the $\hat{z}$ direction.

$$ \hat{\omega} = \omega \sin \psi \hat{x} + \omega \cos \psi \hat{z} $$

And $|V| = \sqrt{R^2 + s^2 - 2R \cos \theta'}$ is independent of $\psi'$, so all integrations involving $\sin \psi'$ or $\cos \psi'$ will be zero, and the only term left is $-\sin \psi \cos \theta' \hat{y}$

For convenience we drop the prime notation

$$ A(\rho) = \frac{\mu_0 \sigma}{4\pi} \int \frac{-R \sin \psi \cos \theta'}{\sqrt{R^2 + s^2 - 2R \cos \theta}} \cdot R^2 \sin \theta \theta d\theta d\psi' $$

Let $u = \cos \theta$, and $du = d\cos \theta = -\sin \theta d\theta$

This means that if $\mathbf{A}'$ is not divergence-free, then we add a $\nabla \lambda$ to it until it is divergence-free. To find out the exact functional form of $\lambda$, we can either solve the Poisson equation at the end of last page, or we can explore the similarity between electrostatic and eq. on page 33.

$$ \lambda = \frac{1}{4\pi} \int \frac{\nabla \cdot \mathbf{A}}{r} \, dr $$

What this means is that Ampere's Law becomes

$$ \nabla \times \mathbf{A} = \mu_0 \mathbf{j} $$

$$ \nabla^2 \mathbf{A} = -\mu_0 \mathbf{j} $$

Coulomb Gauge

We reduce the magnetostatics problem into solving Laplace equation, just like in the electrostatics case.
For \( R < s \) (outside of the shell), equation (1) becomes
\[
\frac{R^2 + s^2 + Rs}{3R^2 s^2} (s - R) - \frac{R^2 + s^2 - Rs}{3R^2 s^2} (s + R) = \frac{2R}{3s^2}
\]
So the vector potential becomes
\[
\vec{A}(P) = -\frac{\mu_0 \sigma R^3 \omega \sin \psi}{2} 2s \hat{Y} \quad \text{(inside)}
\]
\[
\vec{A}(P) = -\frac{\mu_0 \sigma R^3 \omega \sin \psi}{2} \frac{2R}{3s^2} \hat{Y} \quad \text{(outside)}
\]

The magnetic field inside and outside of the shell can be derived from the vector potential \( \vec{A} \)
\[
\vec{B}_{in} = \nabla \times \vec{A}_{in} = \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\theta) - \frac{1}{r} \frac{\partial}{\partial r} (r A_\phi) \hat{\theta}
\]
\[
\vec{B}_{in} = \frac{2 \pi \mu_0 R^2 \omega}{3} \cos \phi - \frac{2 \pi \mu_0 R \omega}{3} \sin \theta \hat{\theta} = \frac{2}{3} \mu_0 R \sigma \hat{z}
\]
Similarly the magnetic field outside can be found
\[
\vec{B}_{out} = \frac{\mu_0 \sigma R^3 \omega}{3r^3} (2 \cos \theta \hat{r} + \sin \theta \hat{\theta})
\]
Compare to eq. 3.103, we can see that it has the form as an \( \vec{E} \) field due to an electric dipole moment.

Summary:
In summary, we show the relationship between source (J), Field (B) and the vector potential (A) in magnetostatics below. It is quite similar to the case of electrostatics.

In Coulomb gauge, we have
\[
\mathbf{E} \cdot \nabla \times \mathbf{A} = \mathbf{0}
\]
\[
\mathbf{B} \cdot \nabla \times \mathbf{A} = \mathbf{0}
\]
In magnetostatics, we have
\[
\mathbf{E} \cdot \nabla \times \mathbf{A} = \mu_0 \mathbf{J}
\]
\[
\mathbf{B} \cdot \nabla \times \mathbf{A} = \mu_0 \nabla \times \mathbf{J}
\]
Since the equations on the left, are almost identical to the equation on the right side, we can use the functional form of Biot-Savart law to express the vector potential as follow.
\[
\vec{A} = \frac{1}{4\pi} \int \frac{\mathbf{j} \times \mathbf{r}'}{r'^2} \, dt'
\]
\[
\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j} \times \mathbf{r}'}{r'^2} \, dt'
\]
(This is true in Coulomb gauge only)

Magnetostatic Boundary conditions
Start with \( \mathbf{E} \cdot \nabla = 0 \), we look at the normal component of the magnetic field across a surface current \( \mathbf{K} \)
\[
\int_{\partial S} \mathbf{B} \cdot d\mathbf{a} = 0
\]
Now start with \( \mathbf{E} \times \mathbf{B} = \mu_0 \mathbf{J} \), and do the line integral show on the figure to the right:
\[
\int_{\partial S} \mathbf{B} \cdot d\mathbf{l} = \mu_0 \mathbf{J} \cdot \mathbf{n} \Rightarrow \mathbf{B}^i_{\text{above}} - \mathbf{B}^i_{\text{below}} = \mu_0 \mathbf{K}
\]
Like the scalar potential in electrostatics, the vector potential is continuous across any boundary, because $\mathbf{V} \cdot \mathbf{A} = 0$ (Coulomb gauge) guarantees that the normal components of $\mathbf{A}$ are continuous, while $\mathbf{V} \times \mathbf{A} = \mathbf{B}$ means that the tangential components of $\mathbf{A}$ are also continuous,

$$\mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}} \quad \text{(right at the boundary)}$$

but

$$\frac{\partial \mathbf{A}_{\text{above}}}{\partial n} - \frac{\partial \mathbf{A}_{\text{below}}}{\partial n} = -\mu \mathbf{j}$$

We can combine the two equations on page 48

$$\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu \mathbf{j} \times \mathbf{R}$$

This can be compared with the equation in electrostatics

$$\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\varepsilon_0} \mathbf{n}$$

The next 3 pages are from Chapter 2, which discuss the boundary conditions across a surface charge distribution $\sigma$

$$\int \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\varepsilon_0}$$

Tangential component of the $\mathbf{E}$ field.

Now we turn our attention to the tangential component of $\mathbf{E}$ field near a surface charge distribution.

$$\int \mathbf{E} \cdot d\mathbf{l} = \mathbf{0}$$

$$\mathbf{E}_{\text{above}} \cdot \mathbf{l} - \mathbf{E}_{\text{below}} \cdot \mathbf{l} = \mathbf{0} \Rightarrow \mathbf{E}_{\text{above}} = \mathbf{E}_{\text{below}}$$

So the parallel component of the $\mathbf{E}$ field is continuous across the surface charges. In general, we can write the boundary condition as follow:

$$\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\varepsilon_0} \mathbf{n} \quad \text{(right at the boundary)}$$

Multipole expansion of the vector potential

From page 33, we can see that the vector potential can be expressed as

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{1}{r'} dV' = \frac{\mu_0}{4\pi} \int \frac{1}{r'} \mathbf{a}' dV'$$

From eq. 3.94, we can expand $\frac{1}{r'}$ in terms of $r$, $r'$ and $P_n(x)$:

$$\frac{1}{r'} = \frac{1}{r} \sum_{n=0}^{\infty} \left( \frac{r}{r'} \right)^n P_n(\cos \theta)$$

Substitute into the expression for $\mathbf{A}$ we end up with

$$\mathbf{A} = \frac{\mu_0 J}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int \left( r' \right)^n P_n(\cos \theta) \cdot dV'$$

Now let’s look at the first few terms of the previous equation.

For $n = 0$,

$$\mathbf{A}_0 = \frac{\mu_0 J}{4\pi} \int \frac{1}{r'} \mathbf{a}' dV'$$

For $n = 1$

$$\mathbf{A}_1 = \frac{\mu_0 J}{4\pi} \int \frac{1}{r^2} r' P_1(\cos \theta) dV'$$

For $n = 2$

$$\mathbf{A}_2 = \frac{\mu_0 J}{4\pi} \int \frac{1}{r^3} r'^2 P_2(\cos \theta) dV'$$

This is multipoles expansion for the vector potential $\mathbf{A}$. 
For \( n = 0 \), it corresponds to the monopole term and
\[
\oint \vec{d}l = 0 \quad \Rightarrow \quad \vec{A}_0 = 0
\]
This implied that magnetic monopole term equals to zero, or magnetic monopole moment does not exit.

For \( n = 1 \), this is the magnetic dipole term and
\[
\vec{A}_1 = \frac{\mu_0 I}{4\pi r^2} \int (r' \cdot \cos \theta) \, d\vec{l}
\]
\[
\vec{A}_1 = \frac{\mu_0 I}{4\pi r^2} \int (\hat{r}' \cdot \hat{r}) \, d\vec{r}.
\]

Note that prime system indicates the source system. Since \( \hat{r} \) is referring to current direction, so \( \hat{r} = \hat{r}' \), but \( l \neq r' \).

Now let “\( d \)” represent “differential” wrt the prime coordinates
\[
\int (\hat{r} \cdot \hat{r}') \, d\vec{r}' = -\int (\hat{r} \cdot \hat{r}') \, d\vec{r}'
\]
and
\[
\hat{r} \times \int (\hat{r} \cdot \hat{r}') \, d\vec{r}' = \int [(\hat{r} \cdot \hat{r}') \, d\vec{r}' - (\hat{r} \cdot \hat{r}') \, d\vec{r}]
\]
From vector triple product rule (Eq. 1.17, page 8).

The magnetic dipole moment \( \vec{m} \) is defined as follow
\[
\vec{m} = \frac{1}{2} \oint (\hat{r} \times \hat{r}') \, d\vec{l} = \frac{1}{2} \oint (\hat{r} \times \hat{r}') \, d\vec{l}
\]
For a planar loop
\[
\frac{1}{2} \oint (\hat{r} \times \hat{r}') \, d\vec{l} = \int d\vec{a} = \vec{a} \vec{a} = \vec{a}
\]
From above equation, we can see that the magnetic dipole moment is independent of the choice of origin. This is not surprising because there is no magnetic monopole.

The Vector potential and magnetic field due to a magnetic dipole located at the origin can be expressed in spherical coordinates
\[
\vec{A}_{\text{dip}} = \frac{\mu_0 m \sin \theta}{4\pi r^2} \hat{r}
\]
\[
\vec{B}_{\text{dip}}(r) = \nabla \times \vec{A}_{\text{dip}} = \frac{\mu_0 m}{4\pi r^3} \left(2 \cos^2 \theta + \sin \theta \theta \right)
\]