CHAPTER 5
MAGNETOSTATICS

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OUTLINES

A brief history of magnetostatics

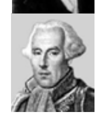
The Chinese compass was invented around 4^{th} century BC.

Compass (circa 4th Century BC)

In 1600, William Gilbert published "De Magnete", one of the first book on electricity and magnetism. Gilbert was regarded by some as the father of electricity and magnetism.

In 1820, Hans Oersted discovered the relationship between electric current and magnetic field, namely a current carrying wire will deflect the needle of a compass.

In the same year, Jean-Baptiste Biot and Felix Savart found that the magnetic field varies inversely with the distance from the wire and curl around the current. The equation describing the magnetic field generated by a current bears their names.



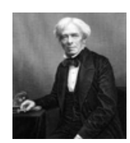
$$\vec{B}(r) = \frac{\mu_o I}{4\pi}\int \frac{\overrightarrow{dl'} \times \hat{r}}{r^2}$$

Ampere learned about Oersted's result in 1820 and started to develop a theory to understand the relationship of electricity and magnetism. Ampere showed that two parallel wires carrying current will attract or repel each other depending on whether the currents are in the same or opposite direction.

In 1826, Ampere discovered the Ampere's law.

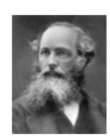
$$\oint \vec{B} \cdot \overrightarrow{dl} = \mu_o I_{enc}$$

Then in 1831, Michael Faraday proposed that change in the magnetic flux can also produce electric field----Faraday's law of induction. The "field line" concept also came from Faraday.



$$\varepsilon = \oint \vec{E} \cdot \overrightarrow{dl} = -\frac{d\Phi}{dt}$$

James Maxwell in 1864 published his Maxwell's equations which combined Gauss's law, Faraday's law, and the modified Ampere's law. Maxwell coined the phrase "the electromagnetic field" for his theory.



Maxwell Equations

- $\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_o}$ $\quad$ $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_o}$
- $\oint \vec{B} \cdot d\vec{a} = 0$ $\quad$ $\nabla \cdot \vec{B} = 0$
- $\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$ $\quad$ $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
- $\oint \vec{B} \cdot d\vec{l} = \mu_o I_{enc} + \mu_o \epsilon_o \frac{d\Phi_E}{dt}$ $\quad$ $\nabla \times \vec{B} = \mu_o j + \mu_o \epsilon_o \frac{d\vec{E}}{dt}$

Let's put things into perspective.

Electrons were discovered by J. J. Thomson in 1897. He found that the cathode rays in a cathode ray tube travel further in air than ions. He estimated that the mass of cathode rays to be at least 1000 times lighter than hydrogen.

In 1887, Michelson and Morley performed the experiment bears their name. They show that there is no "aether" and pave the way for the development of special relativity.

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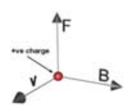
The Lorentz force law

When a charge q moving in a magnetic field $\vec{B}$, it experiences a force called the Lorentz force

$$\vec{F}_m = q \cdot \vec{v} \times \vec{B} \qquad (1)$$



Since the Lorentz force $\vec{F}_m$ is always perpendicular to the velocity, therefore no work done by Lorentz force.

If both E and B are present

$$\vec{F} = q[\vec{E} + (\vec{V} \times \vec{B})]$$

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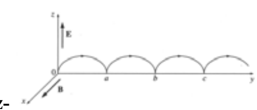
Example 5.2 Cycloid motion

A uniform magnetic field is in the x-direction while another uniform electric field is in the z-direction as shown. Find the trajectory of a charge particle starting at rest at the origin.



Initially at rest at the origin, the charge will experience a force in the z-direction , so initial velocity will be in the z-direction. The Lorentz force will move the charge in the y-direction.

$$\vec{V} = (0, \dot{y}, \dot{z})$$

$$\vec{V} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & \dot{y} & \dot{z} \\ B & 0 & 0 \end{vmatrix} = B\dot{z}\hat{y} - B\dot{y}\hat{z}$$

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$$\vec{F} = q(\vec{E} + \vec{V} \times \vec{B}) = qB\dot{z}\hat{j} + [qE - qB\dot{y}]\hat{k}$$
$$\vec{F} = m\vec{a} = m\ddot{y}\hat{j} + m\ddot{z}\hat{k}$$

$$qB\dot{z} = m\ddot{y}$$
$$qE - qB\dot{y} = m\ddot{z}$$

Let $\frac{qB}{m} = \omega$, the above equations become

$$\ddot{y} = \omega \dot{z} \qquad (1)$$
$$\ddot{z} = \omega\left(\frac{E}{B} - \dot{y}\right) \qquad (2)$$

Take derivative of (1) and then substitute into (2), then integrate over time once, we end up

$$\ddot{y} = \omega^2\left(\frac{E}{B}t - y\right)$$

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The solution to the previous equation can be

$$y(t) = C_1 cos\omega t + C_2 sin\omega t + \frac{E}{B}t + C_3$$

Substitute back into (2), we arrive at

$$\ddot{z} = -\omega^2(C_2 cos\omega t - C_1 sin\omega t)$$

$$z(t) = C_2 cos\omega t - C_1 sin\omega t + C_4$$

Apply the initial conditions, at t = 0, $\vec{r} = 0$, and $\vec{V} = 0$.

$C_1 + C_3 = 0$ and $C_2 + C_4 = 0$

$C_2\omega + \frac{E}{B} = 0$ and $-C_1\omega = 0$

$$C_1 = C_3 = 0 \qquad C_2 = -\frac{E}{B\omega} \qquad C_4 = \frac{E}{B\omega}$$

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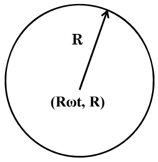
$$y(t) = \frac{E}{B}t - \frac{E}{B\omega}sin\omega t$$

$$z(t) = \frac{E}{B\omega}(1 - cos\omega t))$$

Let $\frac{E}{B\omega} = R$

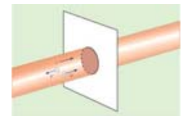
$$(y - R\omega t)^2 + (z - R)^2 = R^2$$

This is the equation for a circle of radius R with the center at (Rωt, R)



Current & Current density

Current in a wire (1D) is defined as the amount of charges passing through a point per unit time.



$I = \frac{Q}{t}$ Ampere = Coulomb per second

If line charge density is λ and velocity is V,

$I = \lambda V$ ⇒ $\vec{I} = \lambda\vec{V}$

Since the current is almost always confined in a conducting wire, we can use the direction of the wire to indicate the direction of current

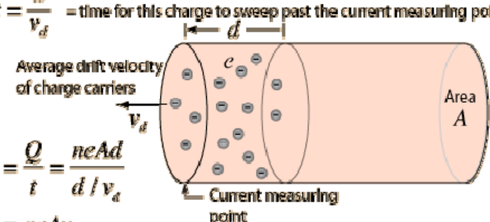
$$\vec{I}dl = Id\vec{l}$$

A 3-D current model (Ideal gas model)

n = number of charges e per unit volume

$Q = neAd$ = total mobile charge in length d of the conductor

$t = \frac{d}{v_d}$ = time for this charge to sweep past the current measuring point.



$$I = \frac{Q}{t} = \frac{neAd}{d / v_d}$$

$$I = neAv_d$$

The concept of Lorentz force on a "charge" can be extended to "current", since "current" can be viewed as "charge density" times velocity.

$$\vec{F}_{mag} = \int(\vec{v} \times \vec{B})dq = \int(\vec{v} \times \vec{B})\lambda dl = \int(\vec{I} \times \vec{B})\,dl$$

Since direction of I is the same as dl, and I is usually a scalar constant,

$$\vec{F}_{mag} = \int I(d\vec{l} \times \vec{B}) = I\int(d\vec{l} \times \vec{B})$$

Example 5.3

A rectangular loop of wire, supporting a mass m, hanging vertically with one end in a uniform magnetic field as shown. What current would be needed to balance the gravitational force?

$F_{mag} = IBa = mg$ ⇒ $I = \frac{mg}{Ba}$

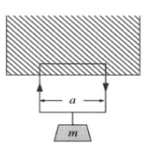


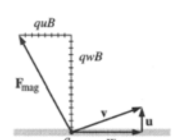
Figure 5.10

What happens if we increase the current?

The magnetic force now will be larger than gravitational force mg, and the mass will rise.

Does that means the magnetic force F_{mag} actual do work?

When the loop starts to rise, the charge no longer moving horizontally, it acquires a vertical component u, so F_m also tilts left as shown. $\vec{F}_m$ is still perpendicular to $\vec{V}$, no work done on charge q. The vertical component, qwB which can be written as

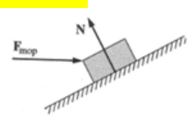


$$F_v = qwB = \lambda awB = IBa$$

The horizontal force, $F_h = quB = \lambda auB$, so the work done by battery to overcome F_h *is*

$$W_{battery} = \lambda aB\int uwdt = IBah$$

This is similar to sliding a block up a frictionless ramp, pushing horizontally



When charge flows on the surface of a conductor (two dimensional), we describe it by the "surface current density" K as

$$\vec{K} = \frac{d\vec{I}}{dl_\perp} \qquad \vec{K} = \sigma\vec{V}$$

$$\vec{F}_{mag} = \int (\vec{K} \times \vec{B}) da$$

When the flow of charge is 3D, we describe it by the volume current density, $\vec{J}$

$$\vec{J} = \frac{\vec{I}}{A} = \frac{Q}{tA} d\hat{l} = \frac{Q \cdot L}{t \cdot A \cdot L} d\hat{l} = \frac{Q}{V} \cdot \frac{L}{t} d\hat{l} = \rho \cdot \vec{V}$$

$$\vec{J} = \rho\vec{V}$$

$$\vec{F}_{Mag} = \int \rho\vec{V} \times \vec{B} d\tau = \int (\vec{J} \times \vec{B}) d\tau$$

From definition of current density

$$I = \int J da_\perp = \int \vec{J} \cdot \overrightarrow{da}$$

They are different

From divergence theorem, we have

$$\oint \vec{J} \cdot \overrightarrow{da} = \int (\nabla \cdot \vec{J}) d\tau$$

From conservation of charge, the amount of charge leaving a volume

$$\oint \vec{J} \cdot \overrightarrow{da} = -\frac{d}{dt} Q_{enc} = -\frac{d}{dt} \int \rho d\tau$$

Combine with previous equation, we have

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

For magneto-statics

$$\nabla \cdot \vec{J} = 0$$

The Biot-Savart Law

Steady current is the source that produce magnetic field in magnetostatics. It plays exactly the same role as stationary charge in electrostatics. The Biot-Savart law plays the exactly same role in magnetostatics as the Coulomb's law which links the source to the field

	Electrostatics	Magnetostatics
Source	Stationary charges	Steady current
Equation	Coulomb's Law	Biot-Savart Law
Field*	Diverge from a point	Curl around a line

Field*---Electric field produced by one point charge, or magnetic field produced by a current on a straight wire.

The magnetic field produced by a steady current,

$$\vec{B}(P) = \frac{\mu_o}{4\pi} \int \frac{\vec{I} \times \hat{r}}{r^2} dl' = \frac{\mu_o I}{4\pi} \int \frac{\overrightarrow{dl'} \times \hat{r}}{r^2}$$

where μ_o is the permeability of the free space.

$$\mu_o = 4\pi \times 10^{-7} \frac{N}{A^2}$$

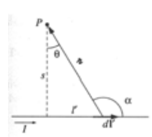
$$\frac{1}{\sqrt{\epsilon_o \mu_o}} = 2.9986 \times 10^8 \ {}^{m}/_{s} \qquad \text{(Speed of light)}$$

The unit of magnetic field $\vec{B}$

$$1 \text{ tesla} = 1 \frac{N}{A \cdot m} \qquad \text{(MKS unit, or SI unit)}$$

$$1\ Gauss = 1 \frac{dyne}{emu \cdot cm} \qquad \text{(cgs unit)}$$

Example 5.5 Find the magnetic field a distance s from a long straight wire carrying a steady current I.



The Magnetic field is given by

$$\vec{B} = \frac{\mu_o}{4\pi} I \int \frac{\overrightarrow{dl'} \times \hat{r}}{r^2}$$

$$\left|\overrightarrow{dl'} \times \hat{r}\right| = dl' \cdot sin\alpha = dl' cos\theta$$

$$l' = s \cdot tan\theta, \qquad and \qquad dl' = s \cdot d(tan\theta) = \frac{s \cdot d\theta}{cos^2\theta}$$

$$\frac{s}{r} = cos\theta \quad \text{and} \quad \frac{1}{r^2} = \frac{cos^2\theta}{s^2}$$

$$B = \frac{\mu_o I}{4\pi} \int \frac{cos^2\theta}{s^2} \cdot \frac{s}{cos^2\theta} d\theta \cdot cos\theta$$

$$B = \frac{\mu_o I}{4\pi s} \int cos\theta \cdot d\theta$$

For infinite straight line,

$$B = \frac{\mu_o I}{2\pi s}$$

Now if we have two long parallel wires carrying current I_1 and I_2, the force between them is

$$\vec{F} = \int I(\overrightarrow{dl} \times \vec{B})$$

$$|\vec{F}| = \left|\int I_2(\vec{dl} \times \vec{B}_1)\right| = \frac{\mu_o I_1 I_2}{2\pi s}\int dl$$

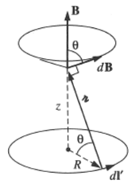
Force per unit length is given by

$$f = \frac{F}{l} = \frac{\mu_o}{2\pi}\left(\frac{I_1 I_2}{s}\right)$$

Example 5.6 Find the magnetic field above the center of a circular loop.

$$B_z(z) = \frac{\mu_o I}{4\pi}\int \frac{dl'}{\mathscr{r}^2} cos\theta$$

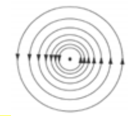
$$B(z) = \frac{\mu_o I}{4\pi}\left(\frac{2\pi R}{R^2+z^2}\right)\frac{R}{\mathscr{r}}$$



The divergence and Curl of $\vec{B}$

For a straight wire carrying a current I, the magnetic field at a distance R is given by

$$B = \frac{\mu_o I}{2\pi}\left(\frac{1}{R}\right)$$



The line integral of this magnetic field is

$$\oint \vec{B}\cdot\vec{dl} = \frac{\mu_o I}{2\pi R}\oint dl = \mu_o I$$

This is Ampere's law. This is true for <u>any shape</u> of closed loop around the current.

We can show that in cylindrical coordinates

$$\vec{B} = \frac{\mu_o I}{2\pi r}\hat{\varphi}, \qquad \vec{dl} = dr\hat{r} + rd\varphi\hat{\varphi} + dz\hat{z}$$

Substitute into the equation below

$$\oint \vec{B}\cdot\vec{dl} = \oint \frac{\mu_o I}{2\pi}\cdot d\varphi = \mu_o I$$

From Stoke's theorem and definition of current density

$$\oint \vec{B}\cdot\vec{dl} = \int(\nabla\times\vec{B})\cdot\vec{da} \qquad I = \int \vec{J}\cdot\vec{da}$$

$\Rightarrow$ $\nabla\times\vec{B} = \mu_o\vec{J}$ Ampere's Law in differential form

Formal derivation of $\nabla\times\vec{B}$ and $\nabla\cdot\vec{B}$

We start out with the Biot-Savart Law

$$\vec{B}(r) = \frac{\mu_o}{4\pi}\int\frac{\vec{J}(r')\times\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau'$$

$$\nabla\cdot\vec{B} = \frac{\mu_o}{4\pi}\int\nabla\cdot\left(\vec{J}(r')\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)d\tau' \qquad \cdots(1)$$

Apply product rule (4) on page 21 of Griffiths to eq. (1) above

$$\nabla\cdot\left(\vec{J}(r')\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot\overset{0}{\cancel{\left(\nabla\times\vec{J}(r')\right)}} - \vec{J}(r')\cdot\overset{0}{\cancel{\left(\nabla\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)}}$$

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$\Rightarrow$ $\nabla\cdot\vec{B} = 0$

Next we'll take the curl of the Biot-Savart law,

$$\nabla\times\vec{B} = \frac{\mu_o}{4\pi}\int\nabla\times\left(\vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)d\tau'$$

The integrant can be re-arranged

$$\nabla\times\left(\vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \vec{J}\overset{4\pi\delta(\mathscr{r})}{\cancel{\left(\nabla\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)}} - (\vec{J}\cdot\nabla)\overset{0}{\cancel{\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}}} + \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\overset{0}{\cancel{(\nabla\cdot\vec{J}(r'))}} - \overset{0}{\cancel{\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot\nabla\right)}}\vec{J}(r')$$

$\Rightarrow$ $\nabla\times\vec{B} = \mu_o\vec{J}$

The second term on the right hand side goes to zero can be seen on next page.

First, we can switch from ∇ to ∇' and change a sign as below

$$-(\vec{J}\cdot\nabla)\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = (\vec{J}\cdot\nabla')\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

Next, we will look at the x-component of the above equation and use integration by parts

$$\nabla'\cdot\left[\frac{(x-x')}{\mathscr{r}^3}\vec{J}\right] = (\vec{J}\cdot\nabla')\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} + \left(\frac{x-x'}{\mathscr{r}^3}\right)\overset{0}{\cancel{(\nabla'\cdot\vec{J})}}$$

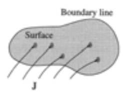
$$\int\nabla'\cdot\left[\frac{(x-x')}{\mathscr{r}^3}\vec{J}\right]d\tau' = \oint\frac{(x-x')}{\mathscr{r}^3}\vec{J}\cdot\vec{da'} = 0$$

(J is zero at large boundary)

Applications of Ampere's Law

The Ampere's law can be written in integral form

$$\oint \vec{B}\cdot\vec{dl} = \mu_o I_{enc}$$

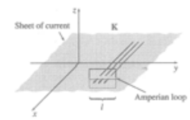


Ampere's Law in magnetostatics plays exactly the same role as the Gauss's Law in electrostatics. But Ampere's law is a line integral while the Gauss's law is a surface integral.

Electrostatics	Coulomb's Law	→ Gauss's Law
Magnetostatics	Biot-Savart Law	→ Ampere's Law

Example 5.8 **Find the magnetic field of an infinite uniform surface current $\vec{K} = k\hat{x}$ flowing over the xy plane.**

The way we solve this problem is very similar to the infinite uniform surface charge problem of Example 2.4 on page 73.



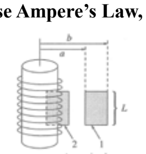
Apply Ampere's Law, we have

$$\oint \vec{B}\cdot\vec{dl} = 2Bl = \mu_o I_{enc} = \mu_o K l$$

$$\vec{B} = \begin{cases} (\mu_o/2)K\hat{y} & for\ \ z<0 \\ -\left(\frac{\mu_o}{2}\right)K\hat{y} & for\ \ z>0 \end{cases}$$

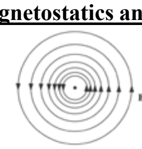
Example 5.9 **Find the magnetic field of a very long solenoid consisting of n closely wound turns per unit length.**

(a) First we argue that there is no magnetic field in the radial direction.
(b) Second we argue that there is no magnetic field in the azimuthal direction (φ).
(c) Third, we argue that outside the solenoid the magnetic field is equal to zero.
(d) Last we argue that the magnetic field inside the solenoid is uniform and in the z-direction.
(e) Use Ampere's Law, we found the field inside to be



$$\vec{B}_{in} = \mu_o n I\hat{z}$$

Magnetostatics and Electrostatics



$$\nabla\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$$

Magnetostatics	Electrostatics
$\nabla\cdot\vec{B} = 0$	$\nabla\cdot\vec{E} = {}^{\rho}/_{\epsilon_o}$
$\nabla\times\vec{B} = \mu_o\vec{J}$	$\nabla\times\vec{E} = 0$
$\vec{B} = \nabla\times\vec{A}$	$\vec{E} = -\nabla V$
$\oint \vec{B}\cdot\vec{dl} = \mu_o I_{enc}$	$\oint \vec{E}\cdot\vec{da} = {}^{Q_{enc}}/_{\epsilon_o}$
$\vec{B} = \frac{\mu_o I}{4\pi}\int \frac{\vec{dl}\times\hat{\mathscr{r}}}{\mathscr{r}^2}$	$\vec{E} = \frac{1}{4\pi\epsilon_o}\int \frac{\rho d\tau}{\mathscr{r}^2}\hat{\mathscr{r}}$

Magnetic Vector Potential

From the fact that $\nabla\cdot\vec{B} = 0$, we can set up a vector potential, $\vec{A}$ such that

$$\vec{B} = \nabla\times\vec{A}$$

To find the expression for $\vec{A}$, we start from Biot-Savart law,

$$\vec{B}(r) = \frac{\mu_o}{4\pi}\int \vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau'$$

$$= \frac{\mu_o}{4\pi}\int \vec{J}\times\left[-\nabla\frac{1}{\mathscr{r}}\right]d\tau'$$

$$\vec{B} = \frac{\mu_o}{4\pi}\int\left[\nabla\times\left(\frac{\vec{J}}{\mathscr{r}}\right) - \frac{1}{\mathscr{r}}\cancelto{0}{\nabla\times\vec{J}}\right]d\tau'$$

Re-arrange the previous equation, we can see that

$$\vec{B} = \nabla\times\left[\frac{\mu_o}{4\pi}\int\frac{\vec{J}}{\mathscr{r}}d\tau'\right]$$

We let the expression in the parenthesis equal to the vector potential of the field. For comparison, electric potential, V(r) is also shown below.

$$\vec{A} = \frac{\mu_o}{4\pi}\int\frac{\vec{J}}{\mathscr{r}}d\tau' \iff V = \frac{1}{4\pi\epsilon_o}\int\frac{\rho}{\mathscr{r}}d\tau'$$

If we add a gradient term to $\vec{A}$, and let $\vec{A}' = \vec{A} + \nabla\lambda$, the magnetic field $\vec{B}$ will remain the same as shown below.

$$\vec{B} = \nabla\times\vec{A}' = \nabla\times\vec{A} + \cancelto{0}{\nabla\times(\nabla\lambda)} = \nabla\times\vec{A}$$

The situation here is similar to the electrostatics case, where add a constant to the potential does not change the electric field at all.

In magnetostatics, adding a gradient term to the vector potential $\vec{A}'$, is called gauge transformation.

$$\vec{A} = \vec{A}' + \nabla\lambda$$

It can be used to choose a particular vector potential. For example, we can choose a vector potential that is divergence-free.

$$\nabla \cdot \vec{A} = 0$$

This will lead to

$$\nabla^2\lambda = -\nabla \cdot \vec{A}'$$

This means that if $\vec{A}'$ is not divergence-free, then we add a $\nabla\lambda$ to it until it is divergence-free. To find out the exact functional form of λ, we can either solve the Poisson equation at the end of last page, or we can explore the similarity between electrostatic and eq. on page 33.

$$\lambda = \frac{1}{4\pi}\int \frac{\nabla \cdot \vec{A}'}{r} d\tau$$

What this means is that Ampere's Law becomes

$$\nabla \times \nabla \times \vec{A} = \mu_o\vec{J} \Rightarrow \nabla(\underbrace{\nabla \cdot \vec{A}}_{0}) - \nabla^2\vec{A} = \mu_o\vec{J}$$

$$\nabla^2\vec{A} = -\mu_o\vec{J}$$
$$\nabla \cdot \vec{A} = 0$$

Coulomb Gauge

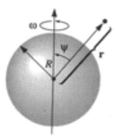
We reduce the magnetostatics problem into solving Laplace equation, just like in the electrostatics case.

Example 5.11 A spherical shell of radius R, carrying a uniform surface charge σ, is set spinning at angular velocity ω. Find the vector potential at point r.

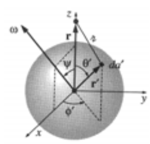
Rotate the axis such that the $\hat{z}$ direction is the same as the $\vec{r}$. The surface current is given by $\vec{K} = \sigma\vec{V}$, and $\vec{V}$ is the velocity of the shell,

$$\vec{V} = \vec{\omega} \times \vec{r'}$$

The $\vec{\omega}$ is pointing at an angle ψ from the $\hat{z}$ direction.



$$\vec{\omega} = \omega sin\psi\hat{x} + \omega cos\psi\hat{z}$$



$$\vec{r'} = Rsin\theta' cos\varphi'\hat{x} + Rsin\theta' sin\varphi'\hat{y} + Rcos\theta'\hat{z}$$

$$\vec{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \omega sin\psi & 0 & \omega cos\psi \\ Rsin\theta' cos\varphi' & Rsin\theta' sin\varphi' & Rcos\theta' \end{vmatrix}$$

$$= R\omega[-(cos\psi sin\theta' sin\varphi')\hat{x} + (cos\psi sin\theta' cos\varphi' - sin\psi cos\theta')\hat{y} + (sin\psi sin\theta' sin\varphi')\hat{z}]$$

The integration is over the surface of the sphere

$$\int da = \int_0^\pi R^2 sin\theta' d\theta' \int_0^{2\pi} d\varphi'$$

And $|\vec{r}| = \sqrt{R^2 + s^2 - 2Rscos\theta'}$ is independent of φ', so all integrations involving $sin\varphi'$ or $cos\varphi'$ will be zero, and the only term left is $-sin\psi cos\theta'\hat{y}$

(for convenience we drop the prime notation)

$$\vec{A}(P) = \frac{\mu_o\sigma}{4\pi}\int \frac{-R\omega sin\psi cos\theta}{\sqrt{R^2 + s^2 - 2Rscos\theta}} \cdot R^2 sin\theta d\theta d\varphi\hat{y}$$

$$= \frac{-\mu_o\sigma R^3\omega sin\psi}{2}\int_0^\pi \frac{cos\theta sin\theta d\theta}{\sqrt{R^2+s^2-2Rscos\theta}}\hat{y}$$

Let $u = cos\theta$, and $du = dcos\theta = -sin\theta d\theta$

So the integral becomes

$$\int_{-1}^{1} \frac{udu}{\sqrt{R^2 + s^2 - 2Rsu}} = -\frac{R^2 + s^2 + Rsu}{3R^2s^2}\sqrt{R^2 + s^2 - 2Rsu}\,\Big|_{-1}^{1}$$

$$= \frac{R^2+s^2+Rs}{3R^2s^2}\sqrt{(R-s)^2} - \frac{R^2+s^2-Rs}{3R^2s^2}\sqrt{(R+s)^2} \qquad (1)$$

For R > s (inside the shell), equation (1) becomes

$$-\frac{R^2 + s^2 + Rs}{3R^2s^2}(R - s) + \frac{R^2 + s^2 - Rs}{3R^2s^2}(R + s) = \frac{2s}{3R^2}$$

For R < s, (outside of the shell), equation (1) becomes

$$\frac{R^2+s^2+Rs}{3R^2s^2}(s-R)-\frac{R^2+s^2-Rs}{3R^2s^2}(s+R)=\frac{2R}{3s^2}$$

So the vector potential becomes

$$\vec{A}(P)=-\frac{\mu_o\sigma R^3\omega sin\psi}{2}\frac{2s}{3R^2}\hat{y} \qquad \text{(inside)}$$

$$\vec{A}(P)=-\frac{\mu_o\sigma R^3\omega sin\psi}{2}\frac{2R}{3s^2}\hat{y} \qquad \text{(outside)}$$

Since $\vec{\omega}\times\vec{s}=-\omega s\cdot sin\psi\hat{y}$, we can re-arrange the equations

$$\vec{A}(P)=\frac{\mu_o\sigma R}{3}(\vec{\omega}\times\vec{s}) \qquad \text{(inside)}$$

$$\vec{A}(P)=\frac{\mu_o\sigma R^4}{3s^3}(\vec{\omega}\times\vec{s}) \qquad \text{(outside)}$$

Now we convert back to the original coordinates with $\vec{\omega}=\omega\hat{k}$, and $\psi=\theta$, and $r=s$, and $\vec{\omega}\times\vec{s}=\omega s sin\theta\hat{\varphi}$

$$\vec{A}(P)=\frac{\mu_o\sigma R\omega}{3}r sin\theta\hat{\varphi} \qquad \text{(inside)}$$

$$\vec{A}(P)=\frac{\mu_o\sigma R^4\omega}{3}\frac{sin\theta}{r^2}\hat{\varphi} \qquad \text{(outside)}$$

The magnetic field inside and outside of the shell can be derived from the vector potential $\vec{A}$

$$\vec{B}_{in}=\nabla\times\vec{A}_{in}=\frac{1}{r sin\theta}\frac{\partial}{\partial\theta}(sin\theta A_{\varphi})\hat{r}-\frac{1}{r}\frac{\partial}{\partial r}(rA_{\varphi})\hat{\theta}$$

$$\vec{B}_{in}=\frac{2\mu_o\sigma R\omega}{3}cos\theta\hat{r}-\frac{2\mu_o\sigma R\omega}{3}sin\theta\hat{\theta} \;=\frac{2}{3}\mu_o R\omega\sigma\hat{z}$$

Similarly the magnetic field outside can be found

$$\vec{B}_{out}=\frac{\mu_o\sigma R^4\omega}{3r^3}(2cos\theta\hat{r}+sin\theta\hat{\theta})$$

Compare to eq. 3.103, we can see that it has the form as an $\vec{E}$ field due to an electric dipole moment.

In Coulomb gauge, we have	In magnetostatics, we have
$\nabla\cdot\vec{A}=0$	$\nabla\cdot\vec{B}=0$
$\nabla\times\vec{A}=\vec{B}$	$\nabla\times\vec{B}=\mu_o\vec{J}$

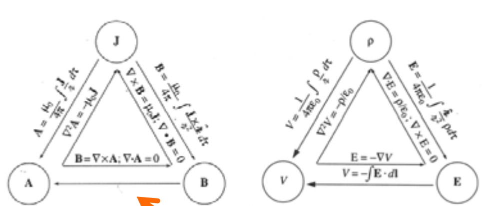
Since the equations on the left, are almost identical to the equation on the right side, we can use the functional form of Biot-Savart law to express the vector potential as follow.

$$\vec{A}=\frac{1}{4\pi}\int\frac{\vec{B}\times\hat{\boldsymbol{r}}}{\boldsymbol{r}^2}d\tau' \qquad \vec{B}=\frac{\mu_o}{4\pi}\int\frac{\vec{J}\times\hat{\boldsymbol{r}}}{\boldsymbol{r}^2}d\tau'$$

(This is true in Coulomb gauge only) Biot-Savart Law

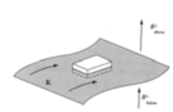
Summary:

In summary, we show the relationship between source (J), Field (B) and the vector potential (A) in magnetostatics below. It is quite similar to the case of electrostatics.



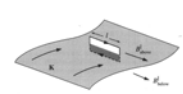
Missing link?

Magnetostatic Boundary conditions



Start with $\nabla\cdot\vec{B}=0$, we look at the normal component of the magnetic field across a surface current $\vec{K}$

$$\oint_s\vec{B}\cdot\vec{da}=0 \quad\Rightarrow\quad B^{\perp}_{above}=B^{\perp}_{below}$$



Now start with $\nabla\times\vec{B}=\mu_o\vec{J}$, and do the line integral show on the figure to the right:

$$\oint\vec{B}\cdot\vec{dl}=\mu_o I_{enc}=\mu_o Kl \quad\Rightarrow\quad B^{\parallel}_{above}-B^{\parallel}_{below}=\mu_o K$$

Like the scalar potential in electrostatics, the vector potential is continuous across any boundary, because $\nabla\cdot\vec{A}=0$ (Coulomb gauge) guarantees that the normal components of $\vec{A}$ are continuous, while $\nabla\times\vec{A}=\vec{B}$ means that the tangential components of $\vec{A}$ are also continuous,

$$\oint\vec{A}\cdot d\vec{l}=\int\vec{B}\cdot d\vec{a}=\Phi \qquad \Phi\Rightarrow 0$$

$$\vec{A}_{above}=\vec{A}_{below} \qquad \text{(right at the boundary)}$$

but
$$\frac{\partial\vec{A}_{above}}{\partial n}-\frac{\partial\vec{A}_{below}}{\partial n}=-\mu_o\vec{K}$$

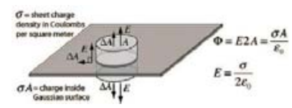
We can combine the two equations on page 48

$$\vec{B}_{above}-\vec{B}_{below}=\mu_o\vec{K}\times\hat{n}$$

This can be compared with the equation in electrostatics

$$\vec{E}_{above}-\vec{E}_{below}=\frac{\sigma}{\epsilon_o}\hat{n}$$

The next 3 pages are from Chapter 2, which discuss the boundary conditions across a surface charge distribution σ



Apply Gauss's Law

$$\oint_{Surface}\vec{E}\cdot\vec{da}=\frac{Q_{enc}}{\epsilon_o}$$

$$\int_A\vec{E}_{above}\cdot\vec{da}+\int_{A'}\vec{E}_{below}\cdot\vec{da}=\frac{\sigma A}{\epsilon_o}$$

$$E^{\perp}_{above}-E^{\perp}_{below}=\frac{\sigma}{\epsilon_o}$$

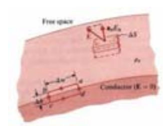
From symmetry argument, the magnitude of the field above and below should be the same, but the directions are different.

$$\vec{E}_{above}=\frac{\sigma}{2\epsilon_o}\hat{k} \qquad \vec{E}_{below}=-\frac{\sigma}{2\epsilon_o}\hat{k}$$

The normal component of the electric field has a "discontinuity" across the charged boundary.

Tangential component of the E field.

Now we turn our attention to the tangential component of E field near a surface charge distribution.



$$\oint\vec{E}\cdot\vec{dl}=0$$

$$=E^{\parallel}_{above}\cdot l-E^{\parallel}_{below}\cdot l=0 \Rightarrow E^{\parallel}_{above}=E^{\parallel}_{below}$$

So the parallel component of the E field is continuous across the surface charges. In general, we can write the boundary condition as follow:

$$\vec{E}_{above}-\vec{E}_{below}=\frac{\sigma}{\epsilon_o}\hat{n} \qquad \text{(right at the boundary)}$$

Multipole expansion of the vector potential

From page 33, we can see that the vector potential can be expressed as

$$\vec{A}=\frac{\mu_o}{4\pi}\int\frac{\vec{J}}{r}d\tau'=\frac{\mu_o I}{4\pi}\oint\frac{1}{r}\vec{dl'}$$

From eq. 3.94, we can expand $\frac{1}{r}$ in terms of r, r' and $P_n(x)$.

$$\frac{1}{r}=\frac{1}{r}\sum_{n=0}\left(\frac{r'}{r}\right)^n P_n(cos\theta)$$

Substitute into the expression for $\vec{A}$ we end up with

$$\vec{A}=\frac{\mu_o I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint(r')^n P_n(cos\theta)\cdot\vec{dl'}$$

Now let's look at the first few terms of the previous equation.

For n = 0,
$$\vec{A}_o=\frac{\mu_o I}{4\pi}\cdot\frac{1}{r}\cdot\oint\vec{dl}$$

For n = 1
$$\vec{A}_1=\frac{\mu_o I}{4\pi}\cdot\frac{1}{r^2}\cdot\oint r'P_1(cos\theta)\vec{dl}$$

For n = 2
$$\vec{A}_2=\frac{\mu_o I}{4\pi}\cdot\frac{1}{r^3}\cdot\oint r'^2P_2(cos\theta)\vec{dl}$$

This is multipole expansion for the vector potential $\vec{A}$.

For n = 0, it is corresponds to the monopole term and

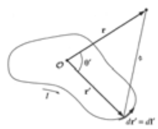
$$\oint \vec{dl} = 0 \quad \Rightarrow \quad \vec{A}_0 = 0$$

This implied that magnetic monopole term equals to zero, or magnetic monopole moment does not exit.

For n = 1, this is the magnetic dipole term and

$$\vec{A}_1 = \frac{\mu_o I}{4\pi r^2} \oint r' \cdot cos\theta \cdot \vec{dl}$$

$$\vec{A}_1 = \frac{\mu_o I}{4\pi r^2} \oint (\vec{r'} \cdot \hat{r})\, \vec{dr'}$$



Note that prime system indicates the source system. Since $\vec{dl}$ is referring to current direction, so $\vec{dl} = \vec{dr'}$, but $l \neq r'$

Now let "d" represent "differential" wrt the prime coordinates

$$d\left[\left(\hat{r} \cdot \vec{r'}\right)\vec{r'}\right] = \left(\hat{r} \cdot \vec{dr'}\right)\vec{r'} + \left(\hat{r} \cdot \vec{r'}\right)\vec{dr'}$$

Since the closed loop integral of any vector equals to zero,

$$\Rightarrow \oint d\left[\left(\hat{r} \cdot \vec{r'}\right)\vec{r'}\right] = 0$$

$$\oint \left(\hat{r} \cdot \vec{dr'}\right)\vec{r'} = -\oint \left(\hat{r} \cdot \vec{r'}\right)\vec{dr'}$$

and

$$\hat{r} \times \oint \left(\vec{r'} \times \vec{dr'}\right) = \oint \left[\left(\hat{r} \cdot \vec{dr'}\right)\vec{r'} - \left(\hat{r} \cdot \vec{r'}\right)\vec{dr'}\right]$$

From vector triple product rule (Eq. 1.17, page 8)

$$\hat{r} \times \oint \left(\vec{r'} \times \vec{dr'}\right) = -2\oint \left(\hat{r} \cdot \vec{r'}\right)\vec{dr'}$$

Since $\vec{dr'} = \vec{dl}$, we will use the results from above and

$$\oint \left(\hat{r} \cdot \vec{r'}\right)\vec{dl} = -\frac{1}{2}\hat{r} \times \oint \left(\vec{r'} \times \vec{dl}\right)$$

Substitute back into eq. on page 55, we have

$$\vec{A}_1 = -\frac{1}{2}\frac{\mu_o I}{4\pi r^2}\hat{r} \times \oint \left(\vec{r'} \times \vec{dl}\right) \quad = \vec{A}_{dip}$$

$$= \frac{\mu_o}{4\pi}\left[\frac{1}{2} I \oint \left(\vec{r'} \times \vec{dl}\right)\right] \times \frac{\hat{r}}{r^2}$$

$$= \frac{\mu_o}{4\pi}\frac{\vec{m} \times \hat{r}}{r^2}$$

As a comparison $V_{dip} = \frac{1}{4\pi\epsilon_o}\frac{\vec{p} \cdot \hat{r}}{r^2}$

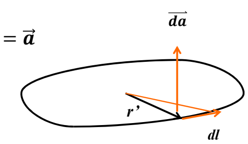
The magnetic dipole moment $\vec{m}$ is defined as follow

$$\vec{m} = \frac{1}{2} I \oint \left(\vec{r'} \times \vec{dl}\right) = \frac{1}{2}\oint \left(\vec{r'} \times \vec{I}\right) dl$$

For a planar loop

$$\frac{1}{2}\oint \left(\vec{r'} \times \vec{dl}\right) = \int \vec{da} = \vec{a}$$

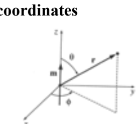
$$\vec{m} = I\vec{a}$$



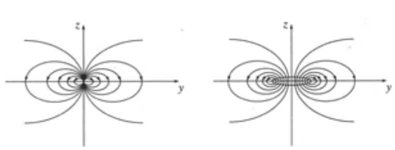
From above equation, we can see that the magnetic dipole moment is independent of the choice of origin. This is not surprising because there is no magnetic monopole.

The Vector potential and magnetic field due to a magnetic dipole located at the origin can be expressed in spherical coordinates

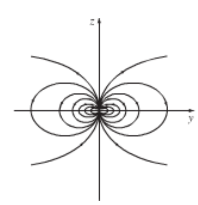
$$\vec{A}_{dip} = \frac{\mu_o}{4\pi}\frac{m\, sin\theta}{r^2}\hat{\phi}$$



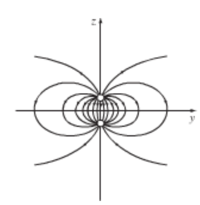
$$\vec{B}_{dip}(r) = \nabla \times \vec{A}_{dip} = \frac{\mu_o m}{4\pi r^3}\left(2cos\hat{r} + sin\theta\hat{\theta}\right)$$



(a) Field of a "pure" dipole (b) Field of a "physical" dipole



Ideal Dipole



Physical Dipole

Electric Dipole