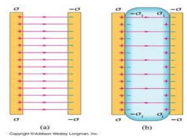
ELECTRIC FIELD IN MATTER



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Outlines

- Polarization
- The field of a polarized object
- The electric displacement, $\vec{D}$
- Linear dielectrics

Polarization

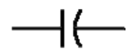
In this chapter we study the effect of an external electric field ($\vec{E}$) on a dielectric material. Dielectric materials are insulators and electrons inside dielectrics are bound to atoms or molecules (bound electrons).

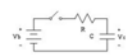
The fundamental concepts in this chapter can be described by the linear response theory. The basic idea is outline below.

Input → **Black box** → Output

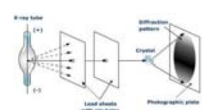
Examples

Resistor  V = IR

Capacitor  $V(t) = \int \frac{I(t)}{C} dt$

RC circuit  $V(t) = V_o e^{-\int \frac{dt}{RC}}$

X-ray scattering



$I(q) = f(q)S(q)$

f(q) is the form factor
S(q) is the structure factor.

For example, we can apply an external magnetic field to a magnetic material such as nickel. The magnetic moment in nickel responded to the external magnetic field by aligning itself to the magnetic field and the net result is magnetization M produced by the aligned magnetic moments.

$$\vec{M} = \chi_m \vec{H}$$

χ_m is the magnetic susceptibility of the material, χ_m is a response function. In the most general case,

$$M_\nu(\vec{k},\Omega) = \sum_q \int d\omega \sum_\mu \chi_{\nu\mu}(\vec{k},\vec{q},\Omega,\omega)\, H_\mu(\vec{q},\omega)$$

Example 4.1: Atom in electric field.

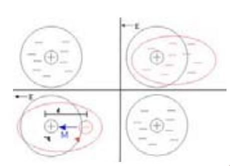
What happens to a neutral atom when it is placed inside an electric field? We assume (1) uniform electron distribution and (2) assume the positive charge only moves a very small amount, so we can ignore it.

The net field on the positive charge is the sum of the pull of the negative charge plus the push of the external field. At equilibrium it equals zero, so:

$$E = \frac{1}{4\pi\epsilon_o}\frac{qd}{a^3}$$

$$p = (4\pi\epsilon_o a^3)E$$

$$p = \alpha E$$



And $\alpha = 4\pi\epsilon_o a^3 = 3\epsilon_o V$ V = volume

For a molecule, there is no symmetry in general, so the polarizability maybe different for different direction.

$$\vec{p} = \alpha_{\perp}\vec{E}_{\perp} + \alpha_{\parallel}\vec{E}_{\parallel}$$

In the most general case in 3D,

$$\begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \alpha_{xx} & \alpha_{xy} & \alpha_{xz} \\ \alpha_{yx} & \alpha_{yy} & \alpha_{yz} \\ \alpha_{zx} & \alpha_{zy} & \alpha_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

We use the lower case $\vec{p}$ to indicate the dipole moment of a single atom or molecule. We use upper case $\vec{P}$ to indicate the polarization of the medium.

$$\vec{P} = \frac{\sum_i \vec{p}_i}{V}$$ Definition of Polarization

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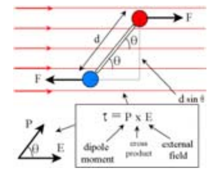
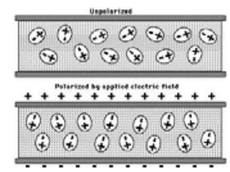
Alignment of polar molecules





There are certain molecules that already have dipole moment. The applied electric field $\vec{E}$ will try to align the dipole moment $\vec{p}$ in the same direction as the field.

$$\vec{p} = q\vec{d}$$

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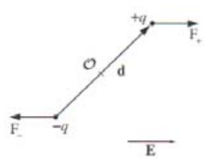
In a uniform electric field $\vec{E}$, the net force on the dipole moment is zero and the net torque $\vec{N}$ on the dipole moment $\vec{p}$ is

$$\vec{N} = (\vec{r}_+ \times \vec{F}_+) + (\vec{r}_- \times \vec{F}_-)$$

$$\vec{N} = \left[\left(\vec{d}/_2\right) \times (q\vec{E})\right] + \left[\left(-\vec{d}/_2\right) \times (-q\vec{E})\right]$$

$$= q\vec{d} \times \vec{E}$$



$$\vec{N} = \vec{p} \times \vec{E}$$

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In a non-uniform field, in addition to the above torque, there will be an additional net force acting on the dipole moment

$$\vec{F} = q(\vec{E}_+ - \vec{E}_-) = q(\Delta\vec{E})$$

This can be re-written as

$$\vec{F} = (\vec{p} \cdot \nabla)\vec{E}$$

Please note that these equations are true even we have perfect ideal dipole moment.

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The field produced by a polarized object

Here we want to find out what is the field (or potential) produced by a dielectric material.
We can treat the dielectrics as material that consists of many dipole moments.

The potential due to ONE ideal dipole moment is

$$V(r) = \frac{1}{4\pi\epsilon_o}\frac{\vec{p}\cdot\hat{\boldsymbol{r}}}{\boldsymbol{r}^2}$$

The potential due to many dipole moments;

$$V(r) = \frac{1}{4\pi\epsilon_o}\int \frac{\vec{P}(r')\cdot\hat{\boldsymbol{r}}}{\boldsymbol{r}^2} d\tau' \quad (4.9)$$

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Since $\nabla'\left(\frac{1}{\boldsymbol{r}}\right) = \frac{\hat{\boldsymbol{r}}}{\boldsymbol{r}^2}$, this leads to

$$V(r) = \frac{1}{4\pi\epsilon_o}\int \vec{P}\cdot\nabla'\left(\frac{1}{\boldsymbol{r}}\right) d\tau'$$

Integrating by part,

$$V(r) = \frac{1}{4\pi\epsilon_o}\int \nabla'\cdot\left(\frac{\vec{P}}{\boldsymbol{r}}\right) d\tau' - \int \frac{1}{\boldsymbol{r}}(\nabla'\cdot\vec{P}) d\tau'$$

Using divergence theorem, the 1st term becomes a surface integral

$$V = \frac{1}{4\pi\epsilon_o}\int \frac{1}{\boldsymbol{r}}\vec{P}\cdot d\vec{a}' - \frac{1}{4\pi\epsilon_o}\int \frac{1}{\boldsymbol{r}}(\nabla'\cdot\vec{P}) d\tau'$$

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Let $\vec{P}\cdot\hat{n} = \sigma_b$ Surface bound charges

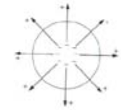
$-\nabla\cdot\vec{P} = \rho_b$ Volume bound charge

The equation at the end of previous page becomes

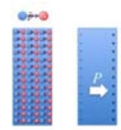
$$V(r) = \frac{1}{4\pi\epsilon_o}\int\frac{\sigma_b}{r}da' + \frac{1}{4\pi\epsilon_o}\int\frac{\rho_b}{r}d\tau'$$

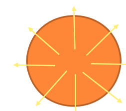
The potential produced by a dielectric object is equivalent to the potential produced by σ_b and ρ_b.

Physical interpretation of bound charges



$\sigma_b = \vec{P}\cdot\hat{n}$ $\rho_b = -\nabla\cdot\vec{P}$





$\vec{P} = P_o\hat{x}$ $\vec{P} = P_o\hat{r}$

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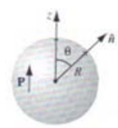
Example 4.2 Find the electric field produced by a uniformly polarized sphere of radius R.



We choose the direction of polarization to be in the z-direction, $\vec{P} = P_o\hat{z}$.

Bound surface charge, $\sigma_b = \vec{P}\cdot\hat{n}$=P$\cos\theta$

Bound volume charge, $\rho_b = -\nabla\cdot\vec{P} = 0$

From example 3.9, equations (3.86) and (3.87)

$V(r,\theta) = \frac{P}{3\epsilon_o}r\cos\theta$ (Inside) r < R

$V(r,\theta) = \frac{PR^3}{3\epsilon_o}\frac{1}{r^2}\cos\theta$ (Outside) r > R

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Once we know the potential, the field can be found.

Inside

$$\vec{E} = -\nabla V = -\frac{\partial V}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\theta}$$

$$\vec{E} = -\frac{P}{3\epsilon_o}\cos\theta\hat{r} + \frac{P}{3\epsilon_o}\sin\theta\hat{\theta} = -\frac{P}{3\epsilon_o}\hat{z} \qquad (4.14)$$

Outside

$$V = \frac{PR^3}{3\epsilon_o}\frac{\cos\theta}{r^2} = \frac{1}{4\pi\epsilon_o}\frac{\frac{4\pi}{3}R^3P}{r^2}\cos\theta$$

$$V = \frac{1}{4\pi\epsilon_o}\frac{\vec{p}_T\cdot\hat{r}}{r^2}$$

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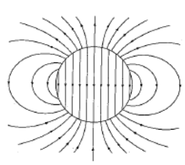
So the potential produced outside the polarized dielectric sphere is equivalent to the field produced by a perfect dipole with a moment that is the sum of all dipole in the sphere. The electric field outside is also the same as the field generated if all dipoles were located at the center of the sphere.

$$E_r = \frac{1}{4\pi\epsilon_o}\frac{2P_T}{r^3}\cos\theta$$

$$E_\theta = \frac{1}{4\pi\epsilon_o}\frac{P_T}{r^3}\sin\theta$$



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Example 4.3 Two uniformly distributed but oppositely charged spheres are slightly displaced such that their centers are separated by a displacement of d as shown. Find the potential due to this charge distribution.

From Problem 2.18, we know that the field at the overlap region is

$$\vec{E} = -\frac{1}{4\pi\epsilon_o}\frac{q\vec{d}}{R^3} \xrightarrow{\vec{P}=\frac{\vec{p}}{V}} \vec{E} = -\frac{1}{3\epsilon_o}\vec{P}$$

Outside the sphere, we can treat this situation as if all charges are concentrated at the respective centers of the spheres:

$$V = \frac{1}{4\pi\epsilon_o}\frac{\vec{p}\cdot\hat{r}}{r^2}$$

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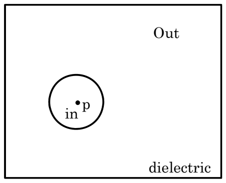
The field inside a dielectric

When we derive the equation to describe the field due to polar molecules and dielectric, we use the ideal dipole formula (slide 11). It is ok to do so when we calculate the field outside of the dielectric. Here we want to show that we can use the same formulism to calculate the electric field even inside the dielectric.

Here we are dealing with <u>macroscopic field</u>, which is an *average field* over a region that contains many thousand of atoms, but still small enough that the concept of differential volume still hold true.



The net electric field at point P, is the sum of the $\vec{E}_{out}$ produced by dipoles outside of the sphere and $\vec{E}_{in}$ produced by dipoles inside the sphere.

$$\vec{E} = \vec{E}_{in} + \vec{E}_{out}$$

<u>Field produced by ouside dipoles</u>

The $\vec{E}_{out}$ is relatively easy, since all dipoles are relatively far away,

$$V_{out} = \frac{1}{4\pi\epsilon_o}\int_{out}\frac{\vec{P}(r')\cdot\hat{r}}{r^2}d\tau'$$

and

$$\vec{E}_{out} = -\nabla V_{out}$$

<u>Field produced by dipoles inside the sphere</u>

For the field due to dipoles inside the sphere, we use the result from problem 3.47, i.e. (eq.3.105)

$$\vec{E}_{ave} = -\frac{1}{4\pi\epsilon_o}\frac{\vec{p}_T}{R^3}$$

where $\vec{p}_T$ is the total dipole moment inside the sphere.

$$\vec{p}_T = \left(\frac{4\pi}{3}R^3\right)\vec{P} \quad \Rightarrow \quad \vec{E}_{in} = -\frac{\vec{P}}{3\epsilon_o}$$

This is the same result (eq. 4.14) derived in Example 4.2 on page 175. So in general, we can integrate the whole space (inside + outside) to find the potential at point p.

$$V = \frac{1}{4\pi\epsilon_o}\int_{all\ space}\frac{\vec{P}\cdot\hat{r}}{r^2}d\tau'$$

The Electric Displacement

We found that the effect of polarization can be "simulated" by assuming bound charges, ρ_b and σ_b generated by $\vec{P}$

$$\rho_b = -\nabla\cdot\vec{P} \qquad \sigma_b = \vec{P}\cdot\hat{n}$$

We call all the other charges the "free charges"

$$\rho_{total} = \rho_b + \rho_f$$

The Gauss's law becomes

$$\epsilon_o\nabla\cdot\vec{E} = \rho = \rho_b + \rho_f = -\nabla\cdot\vec{P} + \rho_f$$

$$\nabla\cdot\left(\epsilon_o\vec{E} + \vec{P}\right) = \rho_f \qquad \text{Let} \qquad \vec{D} = \epsilon_o\vec{E} + \vec{P}$$

$$\nabla\cdot\vec{D} = \rho_f$$

Or in integral form, we have

$$\oint\vec{D}\cdot\vec{da} = Q_{f,enc}$$

However, there is NO Coulomb law for the electric displacement. Also note that the units of $\vec{E}$ and $\vec{D}$ are different.

Physical quantity	Symbol	Unit
Electric Field	$\vec{E}$	Volt/m
Electric displacement	$\vec{D}$	Coulomb/m²
Polarization	$\vec{P}$	Coulomb/m²
Permitivity of free space	ϵ_o	$8.85\times10^{-12}\ C^2N^{-1}m^{-2}$

Problem. 4.15

A spherical shell (hollow sphere)is made of dielectric material with a polarization:

$$\vec{P} = \frac{k}{r}\hat{r}$$

Find the electric field in all three different regions.

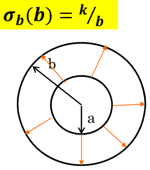
$$\epsilon_o\nabla\cdot\vec{E} = \rho_b + \overset{0}{\cancel{\rho_f}} = -\nabla\cdot\vec{P} \qquad \rho_b = -\frac{k}{r^2}$$

$$\sigma_b(r) = \vec{P}\cdot\hat{n} \qquad \sigma_b(a) = -{}^{k}/_{a}\ , \quad \sigma_b(b) = {}^{k}/_{b}$$

Region I $\quad \vec{E}_I = 0$

Region II $\quad \vec{E}_{II} = -\frac{k}{r\epsilon_o}\hat{r}$

Region III $\quad \vec{E}_{III} = 0$



Region I (inside a)

$$\oint \vec{E} \cdot \overrightarrow{da} = \frac{Q_{enc}}{\epsilon_o} = 0 \Rightarrow \vec{E}_I = 0$$

Region II (inside the dielectric)

$$\oint \vec{E} \cdot \overrightarrow{da} = \frac{Q_{enc}}{\epsilon_o} \qquad Q_{enc} = Q_{surf} + Q_{vol}$$

$$Q_{Surf} = \sigma_b(r = a) \cdot 4\pi a^2 = -4\pi ka$$

$$Q_{Vol} = \int_a^r \rho_b d\tau = \int_a^r \left(-\frac{k}{r^2}\right) 4\pi r^2 dr = -4\pi k(r - a)$$

$$Q_{enc} = (-4\pi ka) + [-4\pi k(r - a)] = -4\pi kr$$

$$\vec{E}_{II} = -\frac{k\,\hat{r}}{\epsilon_o\, r}$$

Region III (r > b)

We will use Gauss's law again. The total enclosed charges have three parts.

$$Q_{enc} = Q_S(r = a) + Q_V + Q_S(r = b)$$

$$Q_S(r = a) = -4\pi ka$$
$$Q_S(r = b) = 4\pi kb$$

$$Q_V = \int_a^b \rho_b \cdot 4\pi r^2 dr = \int_a^b \left(-\frac{k}{r^2}\right) 4\pi r^2 dr$$

$$Q_V = -4\pi k(b - a)$$

$$Q_{enc} = 0 \Rightarrow \vec{E}_{III} = 0$$

Now if we use eq. 4.23,

$$\oint \vec{D} \cdot d\vec{a} = Q_{f_{enc}}$$

Region I

$$\oint \vec{D} \cdot d\vec{a} = Q_{f_{enc}} = 0, \quad \vec{D} = 0 = \epsilon_o \vec{E} + \vec{P}$$

$$\vec{E}_I = -\frac{\vec{P}}{\epsilon_o} = 0$$

Region II

$$\vec{E}_{II} = -\frac{\vec{P}}{\epsilon_o} = -\frac{k\hat{r}}{\epsilon_o r}$$

Region III

$$\vec{E}_{III} = -\frac{\vec{P}}{\epsilon_o} = 0$$

Warnings

(1) No Coulomb law for $\vec{D}$, because $\nabla \times \vec{D}$ maybe non-zero.

$$\vec{D} \neq \int \frac{1}{4\pi} \rho_f \left(\frac{\hat{\boldsymbol{r}}}{\boldsymbol{r}^2}\right) d\tau'$$

(2) $\nabla \times \vec{D} = \epsilon_o(\cancelto{0}{\nabla \times \vec{E}}) + (\nabla \times \vec{P}) = \nabla \times \vec{P}$

(3) No potential can be defined for $\vec{D}$.

Boundary Conditions for E field

$$E^{\parallel}_{Above} - E^{\parallel}_{below} = 0$$

$$E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\epsilon_o}$$

Boundary conditions for displacement, $\vec{D}$

$$D^{\parallel}_{above} - D^{\parallel}_{below} = P^{\parallel}_{above} - P^{\parallel}_{below}$$

$$D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f$$

Linear Dielectrics

For one neutral atom we have

$\vec{p} = \alpha\vec{E}$ α --- polarizability

For dielectric material, we have

$\vec{P} = N\vec{p}$ N is the number of dipoles per unit Volume

If we assume that $\vec{P} \propto \vec{E}$, and let

$\vec{P} = \epsilon_o \chi_e \vec{E}$ Linear dielectric

χ_e is the electric susceptibility

In the most general case,

$$\vec{P} = \epsilon_o[\chi_1 E + \chi_2 E^2 + \chi_3 E^3 + \cdots]$$

$$\vec{D} = \epsilon_o\vec{E} + \vec{P} = \epsilon_o(1+\chi_e)\vec{E}$$

Let $\epsilon = \epsilon_o(1+\chi_e)$

$$\vec{D} = \epsilon\vec{E}$$

We can also define

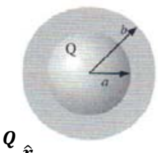
$$\kappa = \frac{\epsilon}{\epsilon_o} = (1+\chi_e)$$

ϵ	Permitivity
κ	Dielectric constant
χ_e	Electric susceptibility

<u>Example</u> 4.5 A metal sphere of radius a carries a charge Q. It is surrounded , out to radius b, by a linear dielectric material of permitivity ε. Find the potential at the center.

We know free charges but not bound charges. Use Gauss's law for $\vec{D}$.

$$\vec{D} = \frac{Q}{4\pi r^2}\hat{r} \qquad \text{For r > a}$$



For b > r > a

$$\vec{E}_{in} = \frac{\vec{D}}{\epsilon} = \frac{1}{4\pi\epsilon}\frac{Q}{r^2}\hat{r}$$

For r > b

$$\vec{E}_{out} = \frac{\vec{D}}{\epsilon_o} = \frac{1}{4\pi\epsilon_o}\frac{Q}{r^2}\hat{r}$$

The potential at the center is:

$$V = -\int_\infty^0 \vec{E}\cdot d\vec{l} = -\int_\infty^b \vec{E}_{out}\cdot d\vec{l} - \int_b^a \vec{E}_{in}\cdot d\vec{l} - \overbrace{\int_a^0 \cancel{\vec{E}_m} d\vec{l}}^{0}$$

$$V = \frac{Q}{4\pi}\left[\frac{1}{\epsilon_o b} + \frac{1}{\epsilon a} - \frac{1}{\epsilon b}\right]$$

(b) Surface bound charges on the dielectric sphere?

$$\vec{P} = \epsilon_o\chi_e\vec{E} = \frac{\epsilon_o\chi_e Q}{4\pi\epsilon r^2}\hat{r} \qquad \rho_b = -\nabla\cdot\vec{P} = -\frac{\epsilon_o\chi_e Q}{4\pi\epsilon}\left(\nabla\cdot\frac{\hat{r}}{r^2}\right)$$

$$\rho_b = 0$$

$$\sigma_b = \vec{P}\cdot\hat{n} = \frac{\epsilon_o\chi_e}{4\pi\epsilon_o}\frac{Q}{b^2} \quad \text{(at r = b)}$$

$$\sigma_b = \vec{P}\cdot\hat{n} = -\frac{\epsilon_o\chi_e}{4\pi\epsilon}\frac{Q}{a^2} \quad \text{(at r = a)}$$

For linear dielectric materials

$$\vec{D} = \epsilon\vec{E}$$

However, this does not imply that $\nabla\times\vec{E} = 0$ will automatically lead to $\nabla\times\vec{D} = 0$ because ϵ can be a function of position.

For a single crystal, the susceptibility can be a tensor.

$$\begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix} = \epsilon_o \begin{pmatrix} \chi_{xx} & \chi_{xy} & \chi_{xz} \\ \chi_{yx} & \chi_{yy} & \chi_{yz} \\ \chi_{zx} & \chi_{zy} & \chi_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

<u>Boundary value problems with linear dielectrics</u>

In a homogeneous linear dielectric

$$\vec{D} = \epsilon_o(1+\chi_e)\vec{E} = \epsilon\vec{E}$$

and

$$\vec{P} = \epsilon_o\chi_e\vec{E}$$

This leads to

$$\vec{P} = \epsilon_o\chi_e\frac{\vec{D}}{\epsilon} = \left(\frac{\chi_e}{1+\chi_e}\right)\vec{D}$$

Assume that χ_e is a constant, take divergence on both side

$$\rho_b = -\left(\frac{\chi_e}{1+\chi_e}\right)\rho_f$$

<u>Boundary conditions between two dielectrics</u>

$\vec{E}_b$ and $\vec{D}_b$
$\vec{E}_a$ and $\vec{D}_a$

<u>Gauss's law for $\vec{D}$</u>

$$D_{b\perp} - D_{a\perp} = \sigma_f$$

$$\epsilon_b E_{b\perp} - \epsilon_a E_{a\perp} = \sigma_f$$

$$If\ \sigma_f = 0$$

$$D_{b\perp} = D_{a\perp}$$

<u>For the E field</u>

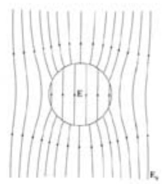
$$\nabla\times\vec{E} = 0$$

$$\oint \vec{E}\cdot d\vec{l} = 0$$

$$E_{a\parallel} = E_{b\parallel}$$

Example 4.7 A sphere of homogeneous linear dielectric material is placed inside an otherwise uniform field, Find the electric field inside the sphere.

This problem can be solved as a boundary value problem. The general solution of Laplace equation is given in eq. 3.65



$$V_{in} = \sum_{l=0} A_l r^l P_l(cos\theta)$$

$$V_{out} = -E_o r cos\theta + \sum_{l=0} \frac{B_l}{r^{l+1}} P_l(cos\theta)$$

The boundary conditions are:

(1) $V_{in} = V_{out}$ at r = R
(2) $\epsilon \frac{\partial V_{in}}{\partial r} = \epsilon_o \frac{\partial V_{out}}{\partial r}$ at r = R
(3) $V_{out} \longrightarrow -E_o r cos\theta$ at r = R

From (1) we have

$$\sum_{l=0} A_l R^l P_l(cos\theta) = -E_o R cos\theta + \sum_{l=0} \frac{B_l}{R^{l+1}} P_l(cos\theta)$$

$$A_l R^l = \frac{B_l}{R^{l+1}} \quad \text{for } l \neq 1$$
$$A_1 R = -E_o R + \frac{B_1}{R^2} \quad \text{for } l = 1$$

From (2) we have (where $\kappa = \epsilon/\epsilon_o$)

$$\kappa \sum_{l=0} l A_l R^{l-1} P_l(cos\theta) = -E_o cos\theta - \sum_{l=0} \frac{(l+1)B_l}{R^{l+2}} P_l(cos\theta)$$

$$\kappa A_l R^{l-1} = -\frac{(l+1)B_l}{R^{l+2}} \quad \text{for } l \neq 1$$
$$\kappa A_1 = -E_o - \frac{2B_1}{R^3} \quad \text{for } l = 1$$

From these two equations and the other two equations from previous page, we conclude that

$$A_l = B_l = 0 \quad \text{for } l \neq 1$$
$$A_1 = -\frac{3}{\kappa+2} E_o \quad \text{and} \quad B_1 = \frac{\kappa-1}{\kappa+2} R^3 E_o$$

$$V_{in} = -\frac{3E_o}{\kappa+2} r cos\theta = -\frac{3E_o}{\kappa+2} z$$

Alternate method (Use eq. 4.30 and iterations)

The basic idea is that the external field $\vec{E}_o$ will induce polarization in the dielectric sphere which in turn will produce extra E field and so on.

$$\vec{E}_o \Rightarrow \vec{P}_o \Rightarrow \vec{E}_1 \Rightarrow \vec{P}_1 \Rightarrow \vec{E}_2 \Rightarrow \vec{P}_2 \Rightarrow$$

$$\vec{P}_o = \epsilon_o \chi_e \vec{E}_o$$

$$\vec{E}_1 = -\frac{1}{3\epsilon_o} \vec{P}_o = \left(-\frac{\chi_e}{3}\right) \vec{E}_o$$

$$\vec{P}_1 = \epsilon_o \chi_e \vec{E}_1 = (\epsilon_o \chi_e) \left(-\frac{\chi_e}{3}\right) \vec{E}_o$$

$$\vec{E}_2 = -\frac{1}{3\epsilon_o} \vec{P}_1 = \left(-\frac{\chi_e}{3}\right)^2 \vec{E}_o$$

$$\vec{E}_{in} = \vec{E}_o + \vec{E}_1 + \vec{E}_2 + \cdots$$

$$\vec{E}_{in} = \left[\sum_{n=0}^{\infty} \left(-\frac{\chi_e}{3}\right)^n\right] \vec{E}_o = \frac{1}{1 + \chi_e/3} \vec{E}_o$$

$$\vec{E}_{in} = \frac{3}{(3+\chi_e)} \vec{E}_o$$

Since $\chi_e + 1 = \kappa$

$$\vec{E}_{in} = \left(\frac{3}{2+\kappa}\right) \vec{E}_o$$

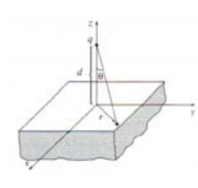
Example 4.8 A charge q is at a distance d away from a semi infinite and uniform linear dielectric material of susceptibility χ_e. Find the surface bound charge, total surface bond charge and force on the point charge.

The surface bound charge density is

$$\sigma_b = \vec{P} \cdot \hat{n} = P_z = \epsilon_o \chi_e E_z \quad (0)$$

E_z is the z-component of E_{in}

E_z has two components, (1) due to charge q, (2) due to σ_b



(1) $-\frac{1}{4\pi\epsilon_o} \frac{q}{(r^2+d^2)} cos\theta = -\frac{1}{4\pi\epsilon_o} \frac{qd}{(r^2+d^2)^{3/2}}$

(2) $-\frac{\sigma_b}{2\epsilon_o}$ (This is inside) why ϵ_o?

So the surface bound charge eq. (0) becomes

$$\sigma_b = \epsilon_o \chi_e \left[-\frac{1}{4\pi\epsilon_o} \frac{qd}{(r^2+d^2)^{\frac{3}{2}}} - \frac{\sigma_b}{2\epsilon_o} \right]$$

$$\sigma_b + \frac{\chi_e \sigma_b}{2} = -\frac{\chi_e}{4\pi} \frac{qd}{(r^2+d^2)^{3/2}}$$

$$\sigma_b = -\frac{1}{2\pi}\left(\frac{\chi_e}{\chi_e+2}\right)\frac{qd}{(r^2+d^2)^{3/2}}$$

Compare with eq. 3.10

$$\sigma_b = \frac{1}{2\pi}\frac{q'd}{(r^2+d^2)^{3/2}}$$

q' can be viewed as an image charge

$$q' = -\left(\frac{\chi_e}{\chi_e+2}\right)q$$

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For z > 0, the potential is given by:

$$V = \frac{1}{4\pi\epsilon_o}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{q'}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

For z < 0, (both charges q and q' are at the same point (0,0,d)

$$V = \frac{1}{4\pi\epsilon_o}\left[\frac{q+q'}{\sqrt{x^2+y^2+(z-d)^2}}\right]$$

The force on the charge q due to the surface charge is

$$F = \frac{1}{4\pi\epsilon_o}\frac{q\cdot q'}{(2d)^2}\hat{k} = -\frac{1}{4\pi\epsilon_o}\left(\frac{\chi_e}{\chi_e+2}\right)\frac{q^2}{(2d)^2}$$

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<u>Example 4.8</u> (Alternate method)

We will use (1) image charges and (2) boundary conditions to solve this problem here.

<u>Boundary condition at z = 0</u>

$E_{in,\parallel} = E_{out,\parallel}$ $D_{in,\perp} = D_{out,\perp}$ $\Rightarrow$ $\epsilon E_{in,\perp} = \epsilon_o E_{out,\perp}$

For z > 0

$$V_{out} = \frac{1}{4\pi\epsilon_o}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{q'}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

For z < 0

$$V_{in} = \frac{1}{4\pi\epsilon_o}\frac{q''}{\sqrt{x^2+y^2+(z-d)^2}}$$

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Let $\rho^2 = x^2 + y^2$

$$E_\parallel = -\frac{\partial V}{\partial \rho}$$

From the b.c. for the E field parallel to the interface, at z = 0

$$-\frac{\partial V_{in}}{\partial \rho} = -\frac{\partial V_{out}}{\partial \rho}$$

$$\Rightarrow \frac{\rho q''}{\sqrt{\rho^2+d^2}^3} = \frac{\rho q}{\sqrt{\rho^2+d^2}^3} + \frac{\rho q'}{\sqrt{\rho^2+d^2}^3}$$

$$\Rightarrow \quad q'' = q + q' \qquad (1)$$

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From the b.c. for the normal $\vec{D}$ at the interface,

$$-\epsilon\frac{\partial V_{in}}{\partial z} = -\epsilon_o\frac{\partial V_{out}}{\partial z}$$

$$\Rightarrow \quad \epsilon q'' = \epsilon_o(q - q') \qquad (2)$$

From (1) and (2)

$$q' = -\frac{\epsilon-\epsilon_o}{\epsilon+\epsilon_o}q \qquad q'' = \frac{2\epsilon_o}{\epsilon_o+\epsilon}q$$

$$q' = -\frac{\epsilon_o(1+\chi_e)-\epsilon_o}{\epsilon_o(1+\chi_e)+\epsilon_o}q = -\frac{\chi_e}{2+\chi_e}q$$

This is the same as derived on page 43.

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<u>Energy in dielectric systems</u>

The amount of energy required to charge a capacitor is

$$dW = V dq = \frac{q}{C}dq$$

$$W = \int dW = \int \frac{q}{C}dq = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2$$

Now if we have a dielectric filling inside the capacitor, the capacitance increases,

$$C' = \kappa C$$

So the work required to charge a capacitor to the same voltage V increases

$$W' = \frac{1}{2}C'V^2 = \kappa\left(\frac{1}{2}CV^2\right)$$

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However, if we only want to charge the capacitor to the same amount of charges, the voltage across the capacitor actually drops

$$V' = \frac{Q}{C'} = \frac{Q}{\kappa C} = \frac{V}{\kappa}$$

And the work done is less

$$W' = \frac{1}{2}C'V'^2 = \frac{1}{2}(\kappa C)\left(\frac{V}{\kappa}\right)^2 = \frac{1}{\kappa}W$$

In Chapter 2, we derive that the energy stored in an electrostatic field is

$$W = \frac{\epsilon_o}{2}\int E^2 d\tau$$

To extend the previous equation to dielectric materials, we re-write the equation as

$$W = \frac{\epsilon}{2}\int E^2 d\tau$$

For linear dielectric materials, we assume that ε is a constant and we can move it into the integral

$$W = \frac{1}{2}\int \vec{D}\cdot\vec{E}d\tau$$

To verify this equation, we argue as follow:

Assume that we bring some free charges near a dielectric, the work done is

$$\Delta W = \int (\Delta\rho_f)V d\tau$$

Since $\nabla\cdot D = \rho_f$

$$\Delta W = \int (\nabla\cdot\Delta\vec{D})V d\tau$$

Integration by part, we have

$$\Delta W = \int \nabla\cdot(\cancel{\Delta\vec{D}V})d\tau + \int \Delta\vec{D}\cdot(-\nabla V)d\tau$$

$$\Delta W = \int \Delta\vec{D}\cdot\vec{E}d\tau$$

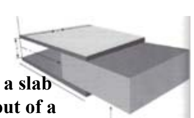
Now if the medium is a linear dielectric, then $\vec{D} = \epsilon\vec{E}$

$$\frac{1}{2}\Delta\left(\vec{D}\cdot\vec{E}\right) = \frac{1}{2}\Delta\left(\epsilon E^2\right) = \epsilon\left(\Delta\vec{E}\right)\cdot\vec{E} = \left(\Delta\vec{D}\right)\cdot\vec{E}$$

$$\Delta W = \Delta\left(\frac{1}{2}\int \vec{D}\cdot\vec{E}d\tau\right)$$

$$W = \frac{1}{2}\int \vec{D}\cdot\vec{E}d\tau$$

Force on dielectrics



Here we study the work done when a slab of linear dielectric material is pull out of a capacitor. This is due to the fringing field around the edge of the capacitor.

$$dW = F_{pull}\cdot dx \quad \Rightarrow \quad F = -\frac{dW}{dx}$$

Capacitance of a half-filled capacitor is,

$$C = \epsilon_o\frac{wx}{d} + \epsilon\frac{w(l-x)}{d} = \epsilon\frac{wl}{d} + \frac{wx}{d}(\epsilon_o - \epsilon)$$

$$C = \epsilon\frac{wl}{d} - \epsilon_o\frac{\chi_e wx}{d}$$

Assuming that total charge on the capacitor is held constant, the force required to pull the dielectric out is

$$F = -\frac{dW}{dx} = -\frac{\partial W}{\partial C}\cdot\frac{dC}{dx} = \frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{dx}$$

From the last equation on previous page,

$$\frac{dC}{dx} = -\frac{\epsilon_o\chi_e w}{d}$$

Substituting back into the above equation

$$F = -\frac{1}{2}V^2\frac{\epsilon_o\chi_e w}{d}$$

Notice that we were able to determine the force without knowing the details of the fringing field.