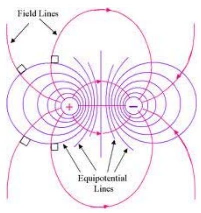
Chapter 3 Potentials



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Outlines

1. Laplace's equation
2. The Method of Images
3. Separation of Variables
4. Multipole Expansion

Laplace's Equation

As we mentioned earlier, in electrostatics the major task is to find $\vec{E}$ field for a given charge distribution. This is basically a "source" problem, can be accomplished by Coulomb's Law and principle of superposition.

$$\vec{E} = \frac{1}{4\pi\epsilon_o}\int \frac{\hat{r}}{r^2}\rho(\vec{r}')d\tau'$$

If the field is too difficult to solve, we can always try to solve the potential

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_o}\int \frac{1}{r}\rho(r')d\tau'$$

In this chapter, we will concentrated on solving the electric potential as a boundary value problem. We start out with Gauss's law in differential form:

$$\nabla\cdot\vec{E} = \frac{\rho}{\epsilon_o} \quad \Longrightarrow \quad \nabla^2 V = -\frac{\rho}{\epsilon_o}$$

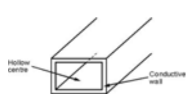
In free space where $\rho = 0$, we have the Laplace's equation.

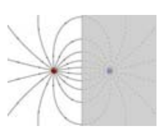
$$\nabla^2 V = 0$$

This type of problems are very common because it is rather easy to set up a boundary condition using a constant voltage power supply.

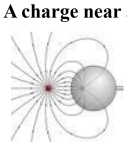
Examples of boundary value problems

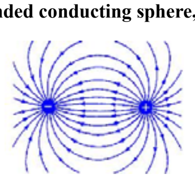
1. An infinite waveguide,





2. A charge near a grounded conducting plane,
3. A charge near a grounded conducting sphere,





4. A dipole.

Laplace's equation in Cartesian coordinates

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

The solution to Laplace's equation are the harmonic functions. The typical method used to solve Laplace's equation is the "separation of variables" technique.

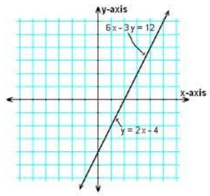
$$V(x,y,z) = V_x(x)\cdot V_y(y)\cdot V_z(z)$$

For the 1D case, we have

$$\frac{d^2V}{dx^2} = 0$$

And the solution can be

$$V(x) = mx + b$$



The above equation has two unknowns, so we will need two boundary conditions to solve this problem. For example: $V(x_1)$ and $V(x_2)$, or $V(x_1)$ and $\frac{dV(x_1)}{dx}$, or $V(x_1)$ and $\frac{dV(x_2)}{dx}$.

Is it possible to have the boundary conditions as $\frac{dV(x_1)}{dx}$ and $\frac{dV(x_2)}{dx}$?

One important thing here is the fact that the solution of Laplace's equation does not have a "local" minimum.

1. For any x, $V(x) = \frac{V(x+a)+V(x-a)}{2}$, $a \rightarrow 0$.
2. Laplace's equation does not allow local minima or maxima. All minima or maxima only occur at the boundary.
3. These properties of the Laplace's equations can be extended to 2D and 3D.

<u>Laplace's equation in two dimension</u>

$$\frac{\partial^2 V(x,y)}{\partial x^2} + \frac{\partial^2 V(x,y)}{\partial y^2} = 0$$

Use separation of variables, we can write V(x, y) as

$$V(x,y) = f(x) \cdot g(y)$$

And substitute back into the Laplace equation above, we end up with

$$g(y) \cdot \frac{\partial^2 f(x)}{\partial x^2} + f(x) \cdot \frac{\partial^2 g(y)}{\partial y^2} = 0$$

$$\frac{1}{f(x)} \cdot \frac{\partial^2 f(x)}{\partial x^2} + \frac{1}{g(y)} \cdot \frac{\partial^2 g(y)}{\partial y^2} = 0$$

In the above equation, since each term depends on one variable only and these two variables are different, so the only way that the above equation is correct is when each term equals to a constant.

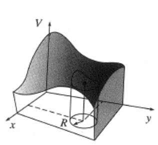
$$\frac{1}{f(x)} \cdot \frac{\partial^2 f(x)}{\partial x^2} = k^2$$

$$\frac{1}{g(y)} \cdot \frac{\partial^2 g(y)}{\partial y^2} = -k^2$$

The solutions to the 2D equation on previous page are harmonic functions. They have the same properties as we noted for the 1D case, i.e.

1. No local maximum or minimum, all extrema occur at the boundary as shown in the figure below.
2. At a given point, the potential is equal to the average potential around that point.

$$V(x,y) = \lim_{R\to 0} \frac{1}{2\pi R} \oint_{circle} V d\ell$$



The radius of the circle is R, and R is small.

The same idea can be extended to 3D

1. No local maximum or minimum, all extrema occur at the boundary as shown in the equation below.
2. At a given point, the potential is equal to the average potential around that point.

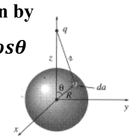
$$V(P) = \lim_{R\to 0} \frac{1}{4\pi R^2} \oint_{sphere} V \cdot da$$

A charge q is located at (0, 0, z), show that the potential at the origin is given by the average potential on the surface of the sphere.

PROOF

Potential on the surface of the sphere is given by

$$V = \frac{1}{4\pi\epsilon_o}\frac{q}{r} \qquad r^2 = z^2 + R^2 - 2zR\cos\theta$$



$$\oint_{sphere} \frac{1}{4\pi\epsilon_o}\frac{q}{4\pi R^2}\frac{R^2 \sin\theta d\theta d\varphi}{\sqrt{z^2 + R^2 - 2zR\cos\theta}}$$

$$= \frac{q}{4\pi R^2}\cdot\frac{R^2}{2\epsilon_o}\cdot\int_0^{2\pi}\frac{\sin\theta d\theta}{\sqrt{z^2 + R^2 - 2zR\cos\theta}}$$

$$= \frac{q}{4\pi R^2}\frac{R^2}{2\epsilon_o}\frac{2}{2zR}[(z+R) - (z-R)] = \frac{1}{4\pi\epsilon_o}\frac{q}{z}$$

The equation on page 13, turn out to be the potential at the origin due to the charge q at (0,0,z), or the potential at point z due to a charge q at the origin:

$$V(P) = \frac{1}{4\pi\epsilon_o}\frac{q}{z}$$

Problem 3.2 Earnshaw's Theorem

Electrostatic force alone can not hold a charge in stable equilibrium, because electrostatic potential satisfied the Laplace's equation, so it cannot have a local minimum.

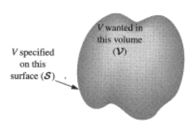
Boundary conditions and uniqueness theorems

For Laplace's equation, if a suitable set of boundary conditions are known, then the solution is uniquely defined.

But what are "suitable" boundary conditions? In 1D, it is rather easy to see. For 2D or 3D cases, the answer to the above questions is presented in the form of "Uniqueness theorems"

First Uniqueness theorem

The solution to Laplace's equation in a volume τ is uniquely determined if the potential V on the boundary surface S is specified.



A hand-waving argument

Assume there are two different solutions, V_1 and V_2 for the same B.C.

Let $V_3 = V_1 - V_2$ and we can see that V_3 is also a solution of the Laplace's equation,

$$\nabla^2 V_3 = \nabla^2(V_1 - V_2) = \nabla^2 V_1 - \nabla^2 V_2 = 0$$

Since the B.C. of V_3 is zero everywhere on the boundary and the solution of Laplace's equation does not allow local minimum, therefore V_3 is zero everywhere inside the boundary.

PROOF

Start with the Green's identity (eq. (c) in Problem 1.61).

$$\int (f\nabla^2 g + \nabla f\cdot\nabla g)d\tau = \oint (f\nabla g)\cdot\overrightarrow{da}$$

Let $U = V_1 - V_2$, and f = U and g = U, it is clear that U = 0 at the boundary and $\nabla^2 U = 0$ inside the volume. So we end up with

$$\int_V (U\nabla^2 U + \nabla U\cdot\nabla U)d\tau = \oint_S (U\nabla U)\cdot\overrightarrow{da}$$

0 0

$\nabla U = 0$ everywhere

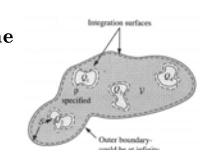
U = constant, everywhere

Since U=0 at boundary, so U=0 everywhere inside the boundary.

The second Uniqueness Theorem

In a volume $\mathcal{V}$, surrounded by conductors and containing a specified charge density ρ, the electric field is uniquely determined if the total charge on each conductor is given.

Assume there are two different E fields satisfied the same B.C.



$$\nabla\cdot\vec{E}_1 = \frac{\rho}{\epsilon_o} \quad \text{and} \quad \nabla\cdot\vec{E}_2 = \frac{\rho}{\epsilon_o}$$

Now let $\vec{E}_3 = \vec{E}_1 - \vec{E}_2$

$$\nabla\cdot\vec{E}_3 = 0 \quad \text{and} \quad \oint \vec{E}_3\cdot\overrightarrow{da} = 0$$

Now let V_3 be the potential that is associated with E_3.

$$\vec{E}_3 = -\nabla V_3$$

Next we will use the product rule,

$$\nabla\cdot(V_3\vec{E}_3) = V_3(\cancelto{0}{\nabla\cdot\vec{E}_3}) + \vec{E}_3\cdot(\nabla V_3) = -(E_3)^2$$

$$\oint_V \nabla\cdot(V_3\vec{E}_3)d\tau = \oint_S V_3\cancelto{0}{\vec{E}_3}\cdot\vec{da} = -\int_V (E_3)^2\,d\tau$$

$$\int_V (E_3)^2 d\tau = 0$$

Since $(E_3)^2$ is always >0 $\Rightarrow$ $\vec{E}_3 = 0$

The third Uniqueness Theorem

If the charge density ρ is given and either V or $\partial V/\partial n$ is given at the boundary, then the $\vec{E}$ field is uniquely determined.

Suppose Let U = $V_1 - V_2$, and $\vec{E}_1 = -\nabla V_1$ and $\vec{E}_2 = -\nabla V_2$

It is clear that $\nabla^2 U = 0$ everywhere and $\partial U/\partial n = 0$ at the boundary. Let's use the Green's identity on page 17.

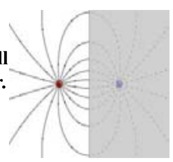
$$\int_V (U\cancel{\nabla^2 U} + \nabla U\cdot\nabla U)\,d\tau = \oint_S (U\cancel{\nabla U})\cdot\vec{da}$$

$$\int_V |\nabla U|^2 d\tau = 0 \Rightarrow \nabla U = 0 \Rightarrow U = \text{const.}$$

$V_1 = V_2 + C$, $\Rightarrow$ $\vec{E}_1 = \vec{E}_2$

The method of images

When a charge is brought near an infinite grounded conducting plane as shown on the right, the electric field will be modified by the grounded conductor. The charge will also induce a surface charge distribution even though the conductor is grounded.



This is simply a "boundary value" problem. We need to find a charge distribution that can "create" the same boundary condition. For a point charge near the infinitely large grounded conducting plane, an image charge on the other side will do the trick.

$$V(x,y,z) = \frac{1}{4\pi\epsilon_o}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{-q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

The electric field is given by

$$\vec{E} = -\nabla V = -\frac{\partial V}{\partial x}\hat{x} - \frac{\partial V}{\partial y}\hat{y} - \frac{\partial V}{\partial z}\hat{z}$$

Surface charge density

$$E_\perp = \frac{\sigma}{\epsilon_o} = -\frac{\partial V}{\partial n} \Rightarrow \sigma = -\epsilon_o \frac{\partial V}{\partial n}\bigg|_{z=0}$$

$$\sigma(x,y,0) = -\left[\frac{1}{4\pi}\left(\frac{-q(0-d)}{\sqrt{x^2+y^2+d^2}^3} - \frac{-q(0+d)}{\sqrt{x^2+y^2+d^2}^3}\right)\right]$$

$$\sigma(x,y,0) = \frac{1}{2\pi}\frac{-qd}{\sqrt{x^2+y^2+d^2}^3}$$

The total charge on the conducting plane is

$$Q = \iint_0^{2\pi} \sigma\cdot rd\theta dr = \iint \frac{-qd\cdot rdr\cdot d\theta}{2\pi\sqrt{r^2+d^2}^3}$$

$$= -qd\left(\frac{-1}{\sqrt{r^2+d^2}}\right)\Bigg|_{r=0}^{r=\infty} = -q$$

Force and Energy

Force between the charge and its image charge.

$$\vec{F} = -\frac{1}{4\pi\epsilon_o}\frac{q^2}{(2d)^2}\hat{k}$$

and

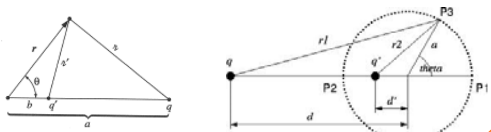
$$W = -\frac{1}{4\pi\epsilon_o}\frac{q^2}{4d}$$ (Only in the space where q is located)

The above equations can be calculated through integration over the whole surface charge.

$$F = \int dF \quad \text{and} \quad W = \int_\infty^d \vec{F}\cdot\vec{dl}$$

A charge near a grounded conducting sphere

When we put two charges q_1 and q_2 a distance 2d apart, if $q_1 = -q_2$, we know that at the mid-point there is a plane that has a zero potential.
Now if $|q_1| \neq |q_2|$, but they do have opposite charges, what will be the shape of the zero potential surface??



The potentials due to q and q' can be written as follow:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}-\vec{r}_1|} + \frac{1}{4\pi\epsilon_o}\frac{q'}{|\vec{r}-\vec{r}'|}$$

where $\vec{r}_1$ is the position of q and $\vec{r}'$ is the position of q'

Now let $\vec{r} = \vec{R}$, and since $V(\vec{R}) = 0$, we end up with

$$\frac{q}{|\vec{R}-\vec{r}_1|} + \frac{q'}{|\vec{R}-\vec{r}'|} = 0$$

1. Choose $\vec{r}'$ to be in the same direction as $\vec{r}_1$, and let $\vec{r}_1 = r_1\hat{a}$ and $\vec{r}' = r'\hat{a}$ (point P1)

2. Let $r' = \frac{R^2}{r_1}$ and $\vec{R} = R\hat{r}$

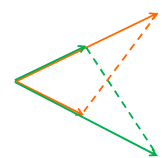
$$\frac{q}{|\vec{R}-r_1\hat{a}|} + \frac{q'}{\left|\vec{R}-\frac{R^2}{r_1}\hat{a}\right|} = \frac{q}{|\vec{R}-r_1\hat{a}|} + \frac{q'}{\frac{R}{r_1}|r_1\hat{r}-R\hat{a}|} = 0$$

3. Let $q' = -\frac{R}{r_1}q$

$$\frac{q}{|R\hat{r}-r_1\hat{a}|} - \frac{q}{|r_1\hat{r}-R\hat{a}|} = 0$$

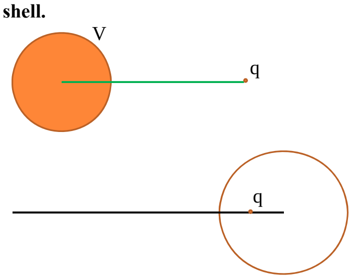
The following expression is always true because of the Law of cosine.

$$|a\hat{a} - b\hat{b}| = |b\hat{a} - a\hat{b}|$$



Therefore, for a charge q near a grounded sphere, we can always find an image charge with $q' = -\frac{R}{r_1}q$, which located at $r' = \frac{R^2}{r_1}$, that will simulate the grounded sphere.

Similar method can be used to find the potential and field of a charge near a conducting sphere with a constant potential of V, or a charge inside a grounded conducting shell.



In Cartesian coordinates, we have

$$V(x,y,z) = \frac{1}{4\pi\epsilon_o}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q\frac{R}{d}}{\sqrt{x^2+y^2+\left(z-\frac{R^2}{d}\right)^2}}\right]$$

In spherical coordinates

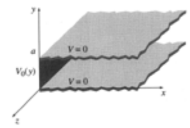
$$V(r,\theta,\phi) = \frac{1}{4\pi\epsilon_o}\left[\frac{q}{\sqrt{r^2+d^2-2rd\cos\theta}} - \frac{q\frac{R}{d}}{\sqrt{r^2+\left(\frac{R^2}{d}\right)^2-2r\frac{R^2}{d}\cos\theta}}\right]$$

$\vec{E}$ field and the surface charge density can be found from the potential V.

Separation of Variables

- Cartesian Coordinates
- Spherical Coordinates
- Cylindrical Coordinates

Separation of variables (Cartesian Coordinates)



B.C. (1) V=0 at y=0
(2) V=0, for $y = a$
(3) V=V(y), x=0
(4) V(x) = 0, as x $\rightarrow\infty$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Let V(x,y)=X(x)· $Y(y)$, we end up with

$$\frac{1}{X(x)} \cdot \frac{\partial^2 X(x)}{\partial x^2} = k^2 \qquad \frac{1}{Y(y)} \cdot \frac{\partial^2 Y(y)}{\partial y^2} = -k^2$$

$$\Rightarrow \quad X(x) = Ae^{kx} + Be^{-kx}$$
$$Y(y) = Csinky + Dcosky$$

$$V(x,y) = \left(Ae^{kx} + Be^{-kx}\right)(Csinky + Dcosky)$$

(1) When y=0, V(x, 0) = 0. $\Rightarrow$ so D = 0.
(2) When y = a, V(x, a) = 0, $\Rightarrow$ sin(ka)=0

ka = nπ, n is integer

(3) When x $\rightarrow \infty$, V = 0, $\Rightarrow$ A = 0.

So we end up with

$$V(x,y) = C'e^{-\frac{n\pi}{a}x} \cdot sin(\frac{n\pi}{a}y) \qquad (C' = CB)$$

Since n can be any integer, we re-write the solution as

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi}{a}x} sin\,(\frac{n\pi}{a}y)$$

Now we match the last boundary condition to find the coefficients C_n.

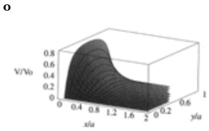
$$V(0,y) = \sum_{n=1}^{\infty} C_n sin\frac{n\pi}{a}y$$

Times both sides by $sin\frac{n'\pi y}{a}$ and integrate from 0 to a

$$C_n \cdot \left(\frac{a}{2}\right) = \int_0^a V(0,y) \cdot sin\left(\frac{n\pi}{a}y\right)dy$$

For example, if V(0,y)= V_o

$$C_n = \frac{2V_o}{a}\int_0^a sin\left(\frac{n\pi}{a}y\right)dy$$



$$= \frac{2V_o}{a} \cdot \frac{a}{n\pi}(1 - cos(n\pi)) = \begin{cases} 0, & \text{for n is even} \\ \frac{4V_o}{n\pi}, & \text{for n is odd} \end{cases}$$

$$V(x,y) = \frac{4V_o}{\pi}\sum_{n=1,3,5}\frac{1}{n}e^{-\frac{n\pi}{a}x} sin\left(\frac{n\pi}{a}y\right)$$

A set of function $f_n(x)$ is said to be "complete" if any function g(x) can be expressed by

$$g(x) = \sum_{n=1}^{\infty} C_n f_n(x)$$

A "complete" set of function has the following properties

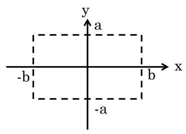
Orthogonal

$$\int f_n(x) f_m(x)dx = 0 \quad \text{if } n \neq m$$

Orthonormal

$$\int f_n(x) \cdot f_m(x)dx = \begin{cases} 0, & n \neq m \\ 1, & n = m \end{cases}$$

Example 3.4 (Modified)



B.C.
1. V=0 at $y = a$
2. V=0 at $y = -a$
3. $V = V_o$ at $x = b$
4. $V = V_o$ at $x = -b$

Similar to example 3.4, but choice of origin is different.

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \qquad \text{let} \qquad V(x,y) = X(x)\cdot Y(y)$$

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = k^2 \qquad \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = -k^2$$

10/13/2016 Chapter 3 Potentials

37

$$X(x) = Ae^{kx} + Be^{-kx}$$
$$Y(y) = C\cdot cos(ky) + D\cdot sin(ky)$$

From B.C. 1. and 2. we have:

$$Y(a) = C\cdot coska + D\cdot sinka = 0$$
$$Y(-a) = C\cdot coska - Dsinka = 0$$

From above we can see that

D = 0, C ≠ 0, and cos(ka) = 0 $\Rightarrow$ $k = \frac{m\pi}{2a}$ m is odd

or

C = 0, D ≠ 0, and sin(ka) = 0 $\Rightarrow$ $k = \frac{m\pi}{a}$ m is integer

10/13/2016 Chapter 3 Potentials

38

Since there is symmetry in the y direction, so the function Y(y) needs to be an even function, therefore

C ≠ 0, D = 0, and $k = \frac{m\pi}{2a}$, m is odd

In the x-direction, X(x) is also an even function, since both e^{kx} and e^{-kx} are not even or odd, we re-construct the solution for X(x) as follow, we find B=0 as follow:

$$X(x) = A\,cosh(kx) + \cancelto{0}{B\,sinh(kx)} = A cosh(kx)$$

A general solution will be

$$V(x,y) = \sum_{m=odd}^{\infty} C_m cosh\left(\frac{m\pi}{2a}x\right) cos\left(\frac{m\pi}{2a}y\right)$$

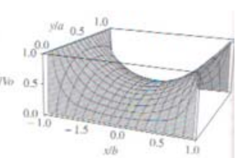
10/13/2016 Chapter 3 Potentials

39

To find the coefficients C_m, we need to use the B.C. 3 and 4 on page 37.

We will leave this as an exercise. Please compare this result with Example 3.4

$$V(x,y) = \frac{4V_o}{\pi}\sum_{m=odd}^{\infty}\frac{1}{m\cdot cosh\left(\frac{m\pi b}{2a}\right)} cosh\left(\frac{m\pi}{2a}x\right) cos\left(\frac{m\pi}{2a}y\right)$$



10/13/2016 Chapter 3 Potentials

40

Spherical coordibates

The Laplace's equation in spherical coordinates is giving below:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2 V}{\partial\varphi^2} = 0$$

Let $V(r,\theta,\varphi) = R(r)\cdot\Theta(\theta)\cdot\Phi(\varphi)$, substitute into the above equation, then divided by $V(r,\theta,\varphi)$

$$\frac{1}{r^2R(r)}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{r^2 sin\theta\cdot\Theta(\theta)}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{r^2 sin^2\theta\cdot\Phi(\varphi)}\frac{\partial^2\Phi}{\partial\varphi^2} = 0$$

The above equation $\times\, r^2 sin^2\theta$

10/13/2016 Chapter 3 Potentials

41

$$\left[\frac{sin^2\theta}{R(r)}\frac{\partial}{\partial r}\left(r^2\frac{\partial R(r)}{\partial r}\right) + \frac{sin\theta}{\Theta(\theta)}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial\Theta}{\partial\theta}\right)\right] + \frac{1}{\Phi(\varphi)}\frac{\partial^2\Phi}{\partial\varphi^2} = 0$$

Since the first two terms are independent of φ, so the last term has to equal to a constant, and let it equal to $-m^2$.

$$\Rightarrow \quad \frac{1}{\Phi(\varphi)}\frac{\partial^2\Phi}{\partial\varphi^2} = -m^2 \qquad (1)$$

The solution of eq. (1) is giving by

$$\Phi(\varphi) = Ae^{+im\varphi} + Be^{-im\varphi}$$

Substitute eq. (1) into the equation at the top, we end up with:

$$\frac{sin^2\theta}{R(r)}\frac{\partial}{\partial r}\left(r^2\frac{\partial R(r)}{\partial r}\right) + \frac{sin\theta}{\Theta(\theta)}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial\Theta(\theta)}{\partial\theta}\right) - m^2 = 0$$

10/13/2016 Chapter 3 Potentials

42

Re-arrange

$$\frac{1}{R(r)}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{1}{\Theta(\theta)\cdot sin\theta}\frac{\partial}{\partial \theta}\left(sin\theta\frac{\partial\Theta}{\partial\theta}\right)-\frac{m^2}{sin^2\theta}=0$$

The 2nd & 3rd terms are independent of r, let them be $-l(l+1)$, after re-arrange:

$$\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\left[l(l+1)-\frac{m^2}{sin^2\theta}\right]\Theta(\theta)=0$$

The 1st term becomes

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)-l(l+1)R(r)=0$$

The solution of this radial equation can be written as

$$R(r)=Ar^l+Br^{-(l+1)}$$

10/13/2016 Chapter 3 Potentials

43

Assume no azimuthal dependence, m = 0, the angular part becomes

$$\frac{1}{sin\theta}\frac{d}{d\theta}\left(sin\theta\frac{d\Theta}{d\theta}\right)+l(l+1)\Theta=0$$

Next, change of variables, we let $cos\theta=x$, then $d(cos\theta)=dx=-sin\theta d\theta$

$$\frac{d}{dx}\left[(1-x^2)\frac{d\Theta(x)}{dx}\right]+l(l+1)\Theta(x)=0$$

The solutions of the above equation are Legendre polynomials of order *l*.

$$\Theta(x)=P_l(x)$$

10/13/2016 Chapter 3 Potentials

44

$$P_0(x)=1$$
$$P_1(x)=x$$
$$P_2(x)=\frac{1}{2}(3x^2-1)$$
$$P_3(x)=\frac{1}{2}(5x^3-3x)$$
$$P_4(x)=\frac{1}{8}(35x^4-30x^2+3)$$

In general, $P_l(x)$ is defined by Rodrigues formula:

$$P_l(x)=\frac{1}{2^l\cdot l!}\left(\frac{d}{dx}\right)^l(x^2-1)^l$$

For m ≠ 0,

$$P_l^m(x)=(-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}(P_l(x))$$

10/13/2016 Chapter 3 Potentials

45

The Legendre polynomial of order *l* is an *l*-th order polynomial.

$P_o(0)=1$, $P_1(0)=0$, $P_2(0)=-1/2$, $P_3(0)=0$, ...

$$P_l(1)=1 \quad \text{for all } l$$

The Legendre polynomials is a complete orthogonal set, namely any function can be expressed using Legendre polynomial. The solution of Laplace equation with azimuthal symmetry can be expressed as

$$V(r,\theta)=\sum_{l=0}^{\infty}\left(A_lr^l+B_lr^{-(l+1)}\right)P_l(cos\theta)$$

10/13/2016 Chapter 3 Potentials

46

The orthogonal properties of $P_l(x)$is given by

$$\int_{-1}^{1}P_{l'}(x)P_l(x)\cdot dx=\frac{2}{2l+1}\delta_{ll'}$$

From Rodrigues formula, we can derive the following recurrence formula.

$$\frac{dP_{l+1}}{dx}-\frac{dP_{l-1}}{dx}-(2l+1)P_l=0$$
$$(l+1)P_{l+1}-(2l+1)xP_l+lP_{l-1}=0$$
$$\frac{dP_{l+1}}{dx}-x\frac{dP_l}{dx}-(l+1)P_l=0$$
$$(x^2-1)\frac{dP_l}{dx}-lxP_l+lP_{l-1}=0$$

10/13/2016 Chapter 3 Potentials

47

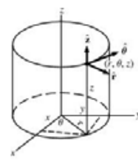
Example 3.6, 3.7 & 3.8

10/13/2016 Chapter 3 Potentials

48

Laplace's equation in cylindrical Coordinates

In cylindrical coordinates, the Laplace's equation is given by



$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Let $V(r,\varphi,z) = R(r)\cdot\Phi(\varphi)\cdot Z(z)$, and substitute in the above equation

$$\frac{1}{rR}\frac{\partial}{\partial r}\left(r\frac{\partial R(r)}{\partial r}\right) + \frac{1}{r^2\Phi}\frac{\partial^2\Phi(\varphi)}{\partial\varphi^2} + \frac{1}{Z(z)}\frac{\partial^2 Z}{\partial z^2} = 0$$

Let the third term equal to a constant

$$\frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2} = k^2 \Rightarrow Z(z) = Ae^{kz} + Be^{-kz}$$

Now go back to the Laplace's equation, we have

$$\frac{1}{rR}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{r^2\Phi}\frac{\partial^2\Phi}{\partial\varphi^2} + k^2 = 0$$

Times r^2, we end up with

$$\frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + k^2r^2 + \frac{1}{\Phi(\varphi)}\frac{\partial^2\Phi}{\partial\varphi^2} = 0$$

Let $\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\varphi^2} = -m^2$, and

$$\Phi(\varphi) = C_m e^{im\varphi} + D_m e^{-im\varphi}$$

The radial component part becomes

$$\frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + k^2r^2 - m^2 = 0$$

$$r^2\frac{\partial^2 R}{\partial r^2} + r\frac{\partial R}{\partial r} + (k^2r^2 - m^2)R = 0$$

Change variable, let $\mathcal{r} = kr$

$$\mathcal{r}^2\frac{\partial^2 R}{\partial \mathcal{r}^2} + \mathcal{r}\frac{\partial R}{\partial \mathcal{r}} + (\mathcal{r}^2 - m^2)R = 0$$

The solution of the above equation is a Bessel function of order m.

If the boundary condition involves an infinite cylinder or wire, such that there is no z-dependence, then k = 0 and the radial function reduces to

$$r^2\frac{d^2R(r)}{dr^2} + r\frac{dR(r)}{dr} - m^2R(r) = 0$$

The solutions to the above equation can be polynomials of r.

$$R(r) = A_m r^m + B_m r^{-m}$$

Combine with the φ-dependence function, we have:

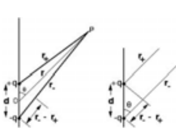
$$V(r,\varphi) = A_o ln(r) + B_o + \sum_{m=1}^{\infty}(A_m r^m + B_m r^{-m})(C_m cosm\varphi + D_m sinm\varphi)$$

Multipole expansion

Here we will explain (a) what is multipoles and (b) show the potential and field of a dipole and a quadrupole. We will also explain (c) what is multipole expansion and why we are interested in multipole expansion.

Electric dipole

A charge q is located at y = d/2, and
A charge –q is located at y = -d/2.



$$V(P) = \frac{1}{4\pi\epsilon_o}\left[\frac{q}{r_+} + \frac{-q}{r_-}\right]$$

Assume that $r \gg d$

$$r_+ = \sqrt{r^2 + \left(\frac{d}{2}\right)^2 - rdcos\theta} \cong r\sqrt{1 - \frac{d}{r}cos\theta}$$

$$r_- = \sqrt{r^2 + \left(\frac{d}{2}\right)^2 - rdcos(\pi - \theta)} \cong r\sqrt{1 + \frac{d}{r}cos\theta}$$

Assume that $r \gg d$

$$r_\pm \approx r\left(1 \mp \frac{d}{2r}cos\theta\right) \Rightarrow \frac{1}{r_+} - \frac{1}{r_-} \cong \frac{d}{r^2}cos\theta$$

Substitute into equation on page 53

$$V(P) \approx \frac{1}{4\pi\epsilon_o}\frac{qd}{r^2}cos\theta$$

Define dipole moment $\vec{P} = q\vec{d}$, where $\vec{d}$ is a displacement vector from –q to q.

$$V(P) = \frac{1}{4\pi\epsilon_o}\frac{\vec{P}\cdot\hat{r}}{r^2}$$

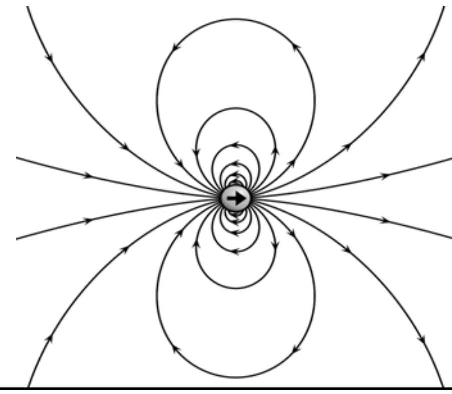
The E-field of a dipole is given by

$$E_r = -\frac{\partial V}{\partial r} = \frac{1}{4\pi\epsilon_o}\frac{2P}{r^3}cos\theta$$

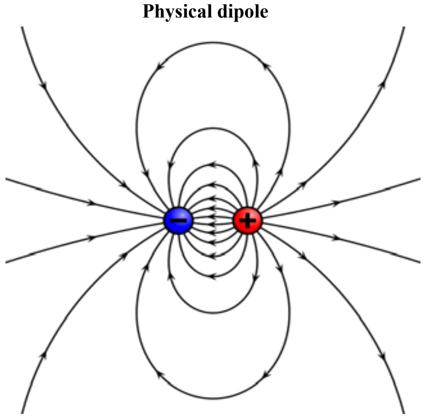
$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{1}{4\pi\epsilon_o}\frac{P}{r^3}sin\theta$$

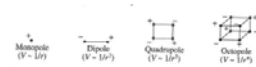
$$E_\varphi = -\frac{1}{rsin\theta}\frac{\partial V}{\partial \varphi} = 0$$

Ideal dipole

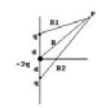


Physical dipole





Linear quadrupole



$$V(r) = \frac{1}{4\pi\epsilon_o}\left[\frac{q}{r_+} - \frac{2q}{r} + \frac{q}{r_-}\right]$$
$$= \frac{q}{4\pi\epsilon_o r}\left[\frac{r}{r_+} + \frac{r}{r_-} - 2\right]$$

$$r_+ = r\sqrt{1 + \left(\frac{d}{r}\right)^2 - \frac{2d}{r}cos\theta}$$

Let $\varepsilon = \left(\frac{d}{r}\right)^2 - \frac{2d}{r}cos\theta$

$$\frac{r}{r_+} = (1+\varepsilon)^{-1/2} = 1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 - \frac{5}{16}\varepsilon^3 + \cdots$$

$$= 1 - \frac{1}{2}\left[\frac{d^2}{r^2} - \frac{2d}{r}cos\theta\right] + \frac{3}{8}\left[\left(\frac{d}{r}\right)^2 - \frac{2d}{r}cos\theta\right]^2 - \cdots$$

$$\cong 1 + \frac{d}{r}cos\theta - \frac{1}{2}\left(\frac{d}{r}\right)^2 + \frac{3}{8}4\left(\frac{d}{r}\right)^2 cos^2\theta + \cdots$$

$$= 1 + \frac{d}{r}cos\theta + \left(\frac{d}{r}\right)^2\left[\frac{3}{2}cos^2\theta - \frac{1}{2}\right] + \cdots$$

Similarly

$$\frac{r}{r_-} \cong 1 - \frac{d}{r}cos\theta + \left(\frac{d}{r}\right)^2\left[\frac{3}{2}cos^2\theta - \frac{1}{2}\right] + \cdots$$

$$V(r) \approx \frac{1}{4\pi\epsilon_o}\frac{2qd^2}{r^3}\left[\frac{3cos^2\theta - 1}{2}\right] + \ldots$$

The monopole term cancels out and the dipole term also cancels out. The potential is proportional to r^{-3}

The electric field of a quadrupole is given by

$$E_r = -\frac{\partial V}{\partial r} = \frac{6qd^2}{4\pi\epsilon_o r^4}\left[\frac{3cos^2\theta}{2}\right]$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{2qd^2}{4\pi\epsilon_o r^4}\cdot 3cos\theta sin\theta$$

We noticed that:

Multipoles	Potential	
monopole	$\sim \frac{1}{r}\cdot 1$	$P_o(cos\theta)$
dipole	$\sim \frac{1}{r^2}\cdot cos\theta$	$P_1(cos\theta)$
quadrupole	$\sim \frac{1}{r^3}\cdot\left[\frac{3}{2}cos^2\theta - \frac{1}{2}\right]$	$P_2(cos\theta)$
Octopole	$\sim \frac{1}{r^4}\cdot\left[\frac{5}{2}cos^3\theta - \frac{3}{2}cos\theta\right]$	$P_3(cos\theta)$

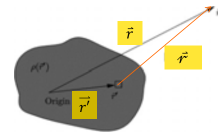
Multipole Expansion

So far we have learned that a point charge (monopole) generates a potential that is $\sim {}^{1}/_{r}$ and +q and –q separated by a small distance creates a dipole potential that is $\sim {}^{1}/_{r^2}$
And if we put two dipole together, we create a quadrupole potential that is $\sim {}^{1}/_{r^3}$ and so on ….

Next we will explain what is multipole expansion and why we want to use this particular technique. We want to show that an arbitrary charge distribution can be expressed in terms of multipole expansion.

$$V(r) = \frac{1}{4\pi\epsilon_o}\int \frac{\rho(r')}{\mathscr{r}}d\tau' \quad \text{and} \quad \vec{\mathscr{r}} = \vec{r} - \vec{r}'$$

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}'$$



$$\mathscr{r} = \sqrt{r^2 + r'^2 - 2rr'\cos\theta}$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\frac{1}{\sqrt{1 + \left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\cos\theta}}$$

Let $\left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\cos\theta = \varepsilon$ ….. (1)

$$\frac{1}{\mathscr{r}} = \frac{1}{r}(1+\varepsilon)^{-1/2}$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\left(1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 - \frac{5}{16}\varepsilon^3 + \cdots\right)$$

Substitute eq. (1) into the last equation

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\left[1 - \frac{1}{2}\left(\left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\cos\theta\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\theta\right)^2 + \cdots\right]$$

Re-arrange

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)\cos\theta + \left(\frac{r'}{r}\right)^2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) + \left(\frac{r'}{r}\right)^3\left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right)\right]$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta)$$

$$V(r) = \frac{1}{4\pi\epsilon_o}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos\theta)\rho(r')d\tau'$$

Let's expand the first few terms,

For n = 0

$$V_0 = \frac{1}{4\pi\epsilon_o}\frac{1}{r}\int \rho(r')d\tau'$$

For n = 1

$$V_1 = \frac{1}{4\pi\epsilon_o}\frac{1}{r^2}\int r' P_1(\cos\theta)\,\rho(r')d\tau'$$

For n = 2

$$V_2 = \frac{1}{4\pi\epsilon_o}\frac{1}{r^3}\int (r')^2 P_2(\cos\theta)\rho(r')d\tau'$$

For n = 3

$$V_3 = \frac{1}{4\pi\epsilon_o}\frac{1}{r^4}\int (r')^3 P_3(\cos\theta)\rho(r')d\tau'$$

So we have shown that an arbitrary charge distribution can be expressed in terms of monopole term, dipole term, quadrupole term, and so on.

In general, the strength of monopole > strength of dipole > strength of quadrupole > strength of octopole

$$S_{mono} > S_{dipole} > S_{quadru} > S_{octo}$$

The dipole term can be written in terms of vectors

$$\frac{1}{4\pi\epsilon_o}\frac{1}{r^2}\int r'\cos\theta\rho(r')d\tau'$$

$$\Rightarrow \frac{1}{4\pi\epsilon_o}\frac{1}{r^2}\hat{r}\cdot\int \vec{r}'\rho(r')d\tau'$$

Define $\vec{P}$ as dipole moment,

$$\vec{P} = \int \vec{r}'\cdot\rho(r')d\tau'$$

$$V_{dipole} = \frac{1}{4\pi\epsilon_o}\frac{\hat{r}\cdot\vec{P}}{r^2}$$

The concept of "moment" comes from mathematics, μ_n is the nth moment of a distribution function f(x)

$$\mu_n = \int x^n \cdot f(x)dx$$

In physics, we define moment of inertia (2nd moment) as

$$I = \int (r')^2\rho(r')d\tau'$$

So we can see that dipole moment is a 1st order moment and quadrupole moment is a 2nd order moment, and so on.

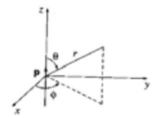
The electric field of a dipole

The electric field of a dipole at the origin derived on slide 55 can be written in the vector form as follow:

$$\vec{E}_{dip}(r,\theta) = \frac{p}{4\pi\epsilon_o r^3}\left(2cos\theta\hat{r} + sin\theta\hat{\theta}\right)$$

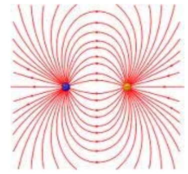
The above equation can be written in the following form

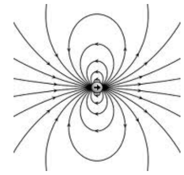
$$\vec{E}_{dip}(\mathbf{r}) = \frac{1}{4\pi\epsilon_o}\frac{1}{r^3}[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}]$$



Physical dipole and ideal dipole

On slide 53, we start out with an expression that describes a physical dipole, we assume that r >> d, we end up with an exact expression of an ideal dipole on slide 55. The ideal dipole expression was re-derived again on slide 65 using multipole expansion. The difference between a physical dipole and an ideal dipole is shown below.





The integrals on slide 62 are for various moments, as we can see that moments will depend on the choice of the origin, except the "monopole moment" which is the net charge. If the net charge is equal to zero, then the next leading term will be independent of the choice of the origin.

So it is meaningless to talk about the dipole moment if (a) net charge is not zero, and (b) we don't know where is the origin.

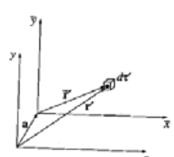
Origin of coordinates in multipole expansion

On slide 64, we showed the first few terms of the multipole expansion. We can see that the monopole term is just the sum of all charges, so it is independent of the coordinates. This is just the charge conservation law. All other higher order terms, such as dipole, quadrupole, or octopole moments will depend on the choice of the origin of the coordinate systems.

In addition, we have a theorem stating that the leading non-zero term of the multipole expansion will always be independent of the choice of the origin of the coordinates. So if the net charge is equal to zero, then the dipole moment of this charge distribution will be independent of the choice of the origin of the coordinate systems. This is demonstrated on the next page:

Assume we have two coordinate systems, the xy system and the $\overline{xy}$ system and

$$\bar{r}' = r' - a$$



So the dipole moment in the $\overline{xy}$ system is given by

$$\bar{p} = \int \bar{r}'\rho(r')d\tau' = \int (r' - a)\rho(r')d\tau'$$

$$= \int r'\rho(r')d\tau' - a\int \rho(r')d\tau' = p - Qa$$

If Q = 0, then $\bar{p} = p$