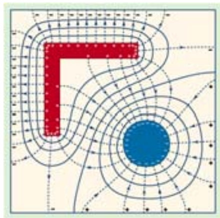
Chapter 2 Electrostatics



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Outlines

1. The electric field
2. $\nabla \cdot \vec{E}$ and $\nabla \times \vec{E}$
3. Electric potential
4. Work and Energy in electrostatics
5. Conductors

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The electric field

The most important concepts in this chapter are:

- Principle of superposition
- Coulomb's law

Superposition theorem

The interaction between any two charges is completely unaffected by the presence of other charges.

Or we can say that if q_1 produces a field of $\vec{E}_1$ and q_2 produces another field $\vec{E}_2$, then the field produced by $q_1 + q_2$ will be $\vec{E}_1 + \vec{E}_2$.

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Superposition theorem may seem "obvious", but however it is not trivial. It originates from the linearity of the field equations. For example, in "non-linear optics", the super position theorem does not hold any true any more.

In general, the topics in E&M can be summarized as: if we know there are a number of charges, q_1, q_2, … , what is the net force they exert on a test charge Q?

In the most general case, each of these charges can have velocity and acceleration and the problem in general is quite difficult to solve, which we will deal with in chapter 9.

Here we only deal with electrostatics.

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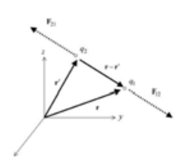
Coulomb's Law

$$\vec{F} = \frac{1}{4\pi\epsilon_o}\frac{qQ}{r^2}\hat{r}$$



Here $\vec{r} = \vec{r} - \vec{r}'$ is the separation vector &
$\epsilon_o = 8.85\times 10^{-12}\frac{C^2}{Nm^2}$ is the permitivity of the free space.

We can remove the test charge Q from the above eq. and define a "field", $\vec{F} = Q\vec{E}$

$$\vec{E} = \frac{1}{4\pi\epsilon_o}\frac{q}{r^2}\hat{r}$$

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The $\vec{E}$ is a field which satisfied the principle of superposition. So the total $\vec{E}$ at point P due to n charges is given by:

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_o}\sum_{i=1}^{n}\frac{q_i}{r_i^2}\hat{r}_i$$

Continuous charge distributions

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_o}\int_L \frac{\lambda(r')}{r^2}\hat{r}\,dl'$$

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_o}\int_S \frac{\sigma(r')}{r^2}\hat{r}\,da'$$

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_o}\int_V \frac{\rho(r')}{r^2}\hat{r}\,d\tau'$$

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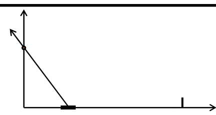
Problem 2.3



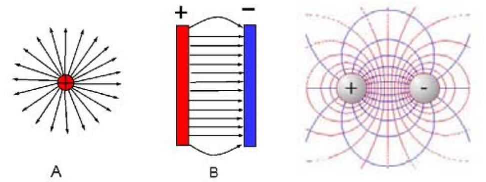
$$dE = \frac{1}{4\pi\varepsilon_o}\frac{\lambda dx}{r^2}$$

$$E_z = \frac{1}{4\pi\varepsilon_o}\int_0^L \frac{\lambda dx}{r^2}\frac{z}{r} = \frac{\lambda z}{4\pi\varepsilon_o}\int_0^L \frac{dx}{(x^2+z^2)^{3/2}}$$

$$E_x = \frac{-1}{4\pi\varepsilon_o}\int_0^L \frac{\lambda dx}{r^2}\frac{x}{z} = \frac{-\lambda}{4\pi\varepsilon_o}\int_0^L \frac{xdx}{(x^2+z^2)^{3/2}}$$

$\nabla \cdot \vec{E}$ and $\nabla \times \vec{E}$

Field lines

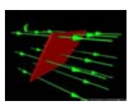


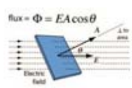
These are electric field lines. They always originated from the positive charges and terminated at negative charges. The density of the lines indicates the magnitude of the electric field.

Flux

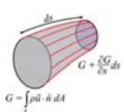
The flux of $\vec{E}$ through a surface S, is defined as

$$\Phi_E \equiv \int_S \vec{E}\cdot\vec{da}$$





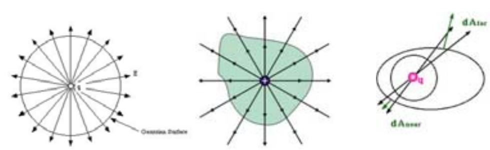
Flux is a measure of the "number of field lines" passing through a surface S. Examples:



For a point charge at the origin, the flux of $\vec{E}$ through a sphere of radius r that contains the origin will be

$$\oint \vec{E}\cdot\vec{da} = \int \frac{1}{4\pi\epsilon_o}\left(\frac{q}{r^2}\hat{r}\right)\cdot\left(r^2 sin\theta d\theta d\varphi\hat{r}\right) = \frac{q}{\epsilon_o}$$

This integration is relatively easy because of symmetry. If the closed surface has an arbitrary shape, will the flux be the same?



Now if we have n charges inside a closed surface, the net field on the surface is given by the principle of superposition:

$$\vec{E} = \sum_{i=1}^{n} \vec{E}_i$$

Substitute into the previous equation, we end up with

$$\oint_S \vec{E}\cdot\vec{da} = \frac{Q_{enc}}{\epsilon_o}$$

Through divergence theorem, we can show that

$$\nabla\cdot\vec{E} = \frac{\rho}{\epsilon_o}$$ **Gauss's Law**

We can also derive the previous equation without using the divergence theorem;

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_o}\int \frac{\hat{\boldsymbol{r}}}{\boldsymbol{r}^2}\rho(r')d\tau'$$

$$\nabla\cdot\vec{E} = \frac{1}{4\pi\epsilon_o}\int\left(\nabla\cdot\frac{\hat{\boldsymbol{r}}}{\boldsymbol{r}^2}\right)\rho(r')d\tau'$$

$$= \frac{1}{4\pi\epsilon_o}\int 4\pi\delta^3(\boldsymbol{r})\rho(r')d\tau'$$

$$= \frac{1}{4\pi\epsilon_o}\int 4\pi\delta^3(\vec{r}-\vec{r}')\rho(r')d\tau'$$

$$= \frac{\rho(\vec{r})}{\epsilon_o}$$

Applications of Gauss's Law

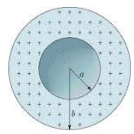
Gauss's Law is a powerful method to find the electric field of certain charge distributions when we can take advantages of symmetry.

Field inside a uniformly charged solid sphere

$$\int_S \vec{E}\cdot\vec{da} = \frac{Q_{enc}}{\epsilon_o}$$

$$E\cdot 4\pi a^2 = \frac{(\frac{4\pi}{3}a^3)\cdot\rho}{\epsilon_o}$$

$$\vec{E} = \frac{a\rho}{3\epsilon_o}\hat{r}$$

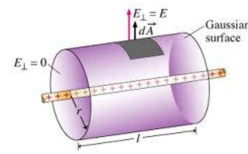


Field near an infinite line charge

$$\oint \vec{E}\cdot\vec{da} = \frac{Q_{enc}}{\epsilon_o}$$

$$E\cdot 2\pi r\cdot l = \frac{\lambda\cdot l}{\epsilon_o}$$

$$\vec{E} = \frac{1}{2\pi\epsilon_o}\frac{\lambda}{r}\hat{r}$$

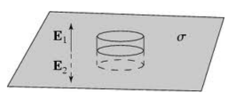


E field near a uniform 2D surface charge

$$\oint \vec{E}\cdot\vec{da} = \frac{Q_{enc}}{\epsilon_o}$$

$$2E\cdot\pi R^2 = \frac{\pi R^2\cdot\sigma}{\epsilon_o}$$

$$\vec{E} = \frac{\sigma}{2\epsilon_o}\hat{n}$$



The Curl of $\vec{E}$

From Maxwell Equation,

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

For electrostatic, there is no time-dependent terms, therefore the curl of a static $\vec{E}$ is zero everywhere.

$$\nabla\times\vec{E} = 0$$

The above result can be obtained directly assuming that the $\vec{E}$ field produced by a point charge at the origin.

So static E field is a curl-less field.

$$\vec{E} = \frac{1}{4\pi\epsilon_o}\frac{q}{r^2}\hat{r} = E_r\hat{r}$$

$$\nabla\times\vec{E} = \frac{q}{4\pi\epsilon_o}\nabla\times\frac{\hat{r}}{r^2}$$

$$= \frac{q}{4\pi\epsilon_o}\left[\frac{1}{r\cdot sin\theta}\left[\frac{\partial}{\partial\theta}\left(sin\theta\cdot E_\varphi - \frac{\partial E_\theta}{\partial\varphi}\right)\right]\hat{r}\right]$$

$$+\frac{1}{r}\left[\frac{1}{sin\theta}\frac{\partial E_r}{\partial\varphi} - \frac{\partial}{\partial r}(rE_\varphi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rE_\theta) - \frac{\partial E_r}{\partial\theta}\right]\hat{\varphi}$$

$$= 0$$

All central fields are curl-less fields.

We can also calculate the line integral of the Coulomb field as follow:

$$\int_a^b \vec{E}\cdot\vec{dl} = \frac{1}{4\pi\epsilon_o}\int_a^b \frac{q}{r^2}dr = \frac{-1}{4\pi\epsilon_o}\frac{q}{r} = \frac{q}{4\pi\epsilon_o}\left(\frac{1}{a} - \frac{1}{b}\right)$$

For a closed path,

$$\oint \vec{E}\cdot\vec{dl} = 0 \quad \Rightarrow \quad \nabla\times\vec{E} = 0$$

For arbitrary charge distribution, $\vec{E}_{total} = \vec{E}_1 + \vec{E}_2 + \cdots$

$\nabla\times\vec{E}_{total} = 0$ If E_i are statics fields

What is the consequence of a Curl-free vector field?

- It is not an arbitrary field, you can not write down an arbitrary function and call it an electric field.
- It can be expressed as a gradient of a scalar field,

$$\vec{E} = -\nabla V$$

where V is a scalar field called potential. For a vector field that we can associate a "potential" to the field, the field is a conservative field.

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The line integral of $\vec{E}$ field is independent of the path, we can choose a reference point, such that the line integral from a reference to a point $\vec{r}$ is given by:

$$-\int_{ref}^{a} \vec{E}\cdot\vec{dl} = V(a) \qquad -\int_{ref}^{b} \vec{E}\cdot\vec{dl} = V(b)$$

$$V(b) - V(a) = -\int_{ref}^{b} \vec{E}\cdot\vec{dl} + \int_{ref}^{a} \vec{E}\cdot\vec{dl}$$

$$= -\int_a^b \vec{E}\cdot\vec{dl} = \int_a^b (\nabla V)\cdot\vec{dl}$$

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From previous equation, we can see that

$$\vec{E} = -\nabla V$$

The concept of "path independence" is very important here. It allows us to define a potential and it gives us a "conservative field".

V is called potential and it is a scalar function. If we know potential V, we can find the field by taking the gradient of the potential. There are two advantages of using V to find $\vec{E}$.

- V is a scalar, but $\vec{E}$ is a vector
- The equations for V is 2nd order DE, while equations for $\vec{E}$ are 1st order DE.

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The field is a vector, it seems to contain much more information than the potential, which is scalar function. In reality, there are a lot of redundant information contained in the field, because the static electric field is a curl-free field.

The choice of reference point in the potential is arbitrary. So only the potential difference is important. WE can always add a constant to a potential and it will not affect the electric field $\vec{E}$ at all.

Typically when we have a finite charge distribution, we choose $r = \infty$ as our reference point where the potential is zero.

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Example 7 and Problem 21 will be shown in class

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Poisson's and Laplace's Equations

Since we can write down the electric field as the negative of the gradient of a potential:

$$\vec{E} = -\nabla V$$

The fundamental equation for $\vec{E}$ becomes

$$\nabla\cdot\vec{E} = \frac{\rho}{\epsilon_o} \quad \Rightarrow \quad \nabla^2 V = -\frac{\rho}{\epsilon_o}$$

For $\rho = 0$, we have a Laplace's eq.

$$\nabla^2 V = 0$$

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So for V, we have only one 2nd order DE to solve, but if we approach the problem using electric field $\vec{E}$, we end up with two equations:

$$\nabla \cdot \vec{E} = 0, \qquad \nabla \times \vec{E} = 0$$

In general, there are two major ways to solve the potential problems in electrostatic:

(a) Solve as a source problem using integration,
(b) Solve as a boundary value problem, using boundary conditions.

In the following, we will start out using method (a). Method (b) will be discussed in Chapter 3.

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The potential of a localized charge distribution

We want to calculate the potential if given a fixed charge distribution, $\rho(\vec{r})$:

For a point charge, we know that $\vec{E} = \frac{1}{4\pi\epsilon_o}\frac{q}{r^2}\hat{r}$

$$V = -\int_{\infty}^{r} \vec{E} \cdot \vec{dl} = -\int_{\infty}^{r} E \cdot dr = -\frac{q}{4\pi\epsilon_o}\int_{\infty}^{r} \frac{dr}{r^2}$$

$$= \frac{q}{4\pi\epsilon_o} \cdot \frac{1}{r}\Big\{_{\infty}^{r} = \frac{1}{4\pi\epsilon_o}\frac{q}{r}$$

If the source is not at the origin,

$$V = \frac{1}{4\pi\epsilon_o} \cdot \frac{q}{r}$$

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For a continuous 3D charge distribution,

$$V(r) = \frac{1}{4\pi\epsilon_o}\int \frac{\rho(r')}{r} d\tau'$$

For a surface charge distribution

$$V(r) = \frac{1}{4\pi\epsilon_o}\int \frac{\sigma(r')}{r} da'$$

For a line charge distribution

$$V(r) = \frac{1}{4\pi\epsilon_o}\int \frac{\lambda(r')}{r} dl'$$

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Example 8 and Problem 25

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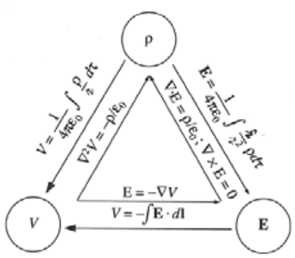
Summary: Electrostatic boundary conditions



Relationship between (a) charge density, (b) potential, and (c) electric field.

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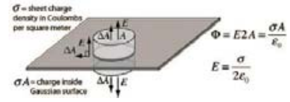
From example 2-7, we can see that if we cross a surface charge density σ, the potential is continuous , but the E field has a dis-continuity across the boundary.

Now we want to explore the usage of "Gauss's Law" to solve electrostatic problem.



Apply Gauss's Law

$$\oint_{Surface} \vec{E} \cdot \vec{da} = \frac{Q_{enc}}{\epsilon_o}$$

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$$\int_A \vec{E}_{above} \cdot \overrightarrow{da} + \int_{A'} \vec{E}_{below} \cdot \overrightarrow{da} = \frac{\sigma A}{\epsilon_o}$$

$$E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\epsilon_o}$$

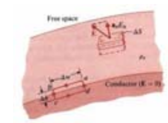
From symmetry argument, the magnitude of the field above and below should be the same, but the directions are different.

$$\vec{E}_{above} = \frac{\sigma}{2\epsilon_o}\hat{k} \qquad \vec{E}_{below} = -\frac{\sigma}{2\epsilon_o}\hat{k}$$

The normal component of the electric field has a "discontinuity" across the charged boundary.

Tangential component of the E field.

Now we turn our attention to the tangential component of E field near a surface charge distribution.



$$\oint \vec{E} \cdot \overrightarrow{dl} = 0$$

$$= E^{\parallel}_{above} \cdot l - E^{\parallel}_{below} \cdot l = 0 \Rightarrow E^{\parallel}_{above} = E^{\parallel}_{below}$$

So the parallel component of the E field is continuous across the surface charges. In general, we can write the boundary condition as follow:

$$\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\epsilon_o}\hat{n}$$

Potential

The potential is continuous across the boundary because

$$\int_a^b \vec{E} \cdot \overrightarrow{dl} = -[V(b) - V(a)]$$

$$V_{above} = V_{below}$$

Since $\vec{E} = -\nabla V$,

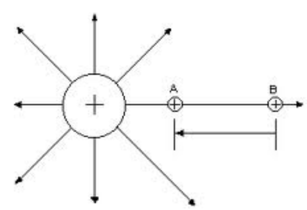
$$\nabla V_{above} - \nabla V_{below} = -\frac{\sigma}{\epsilon_o}\hat{n}$$

Times the normal unit vector on both sides:

$$\frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\frac{\sigma}{\epsilon_o}$$

Work and energy in electrostatics

Assume that we want to move a charge Q from point B to point A. What is the "work" done?



The force needed to overcome the Coulomb force is given by,

$$\vec{F} = -Q\vec{E}$$

The work done by this force from B to A is

$$W = \int_B^A \vec{F} \cdot \overrightarrow{dl} = -Q\int_B^A \vec{E} \cdot \overrightarrow{dl}$$

$$= Q\int_B^A \nabla V \cdot \overrightarrow{dl} = Q(V(A) - V(B))$$

$$= Q \cdot \Delta V$$

So the work done is equal to the amount of charge times the "potential difference" between those two points. Now if we want to move a charge from infinity to a point P, then

$$W = Q[V(P) - V(\infty)]$$

If we assume that V(∞) = 0 as our reference point

$$W = Q \cdot V(P)$$

where V(P) is the potential at point P due to all q_i charges.

Now let's find out the energy of a charge collection.

1. Put the first charge q_1 at position r_1
   No work done.
2. Put the 2nd charge at position r_2
   work done equal to

$$W_2 = q_2 \cdot \frac{1}{4\pi\epsilon_o} \frac{q_1}{|\vec{r}_1 - \vec{r}_2|}$$

3. Put a third charge into the potential created by q_1 and q_2.

$$W_3 = q_3 V_1 + q_3 V_2$$

$$= \frac{1}{4\pi\epsilon_o} q_3 \left[\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}}\right]$$

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The energy required to add 4th charge into the system is

$$W_4 = q_4[V_1 + V_2 + V_3]$$

$$= \frac{1}{4\pi\epsilon_o} q_4 \left[\frac{q_1}{r_{14}} + \frac{q_2}{r_{24}} + \frac{q_3}{r_{34}}\right]$$

For N charges

$$W_{Total} = \frac{1}{4\pi\epsilon_o} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j>i}}^{N} \frac{q_i q_j}{r_{ij}}$$

$$= \frac{1}{8\pi\epsilon_o} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{q_i q_j}{r_{ij}} = \frac{1}{2} \sum_{i=1}^{N} q_i V(P_i)$$

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For continuous charge distribution,

$$W = \frac{1}{2} \int_\tau \rho \cdot V \cdot d\tau$$

We can express the charge density in terms of the divergence of the E field.

$$W = \frac{1}{2} \epsilon_o \int (\nabla \cdot \vec{E}) V d\tau$$

Integration by part

$$W = \frac{\epsilon_o}{2} \left[\int \nabla \cdot (\vec{E}V) d\tau - \int \vec{E} \cdot \nabla V d\tau \right] = \frac{\epsilon_o}{2} \int E^2 d\tau$$

$$\int_S (V\vec{E}) \cdot \vec{da} = 0$$

Assume finite charge distribution

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Example 8 and Problem 2.32(b)

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Comments on electrostatic energy

1. Difference between Equation 2.42 and equation 2.45

$$W = \frac{1}{2} \sum_{i=1}^{N} q_i V(\vec{r}_i) \qquad (2.42)$$

$$W = \frac{1}{2} \int \rho V \cdot d\tau = \frac{\epsilon_o}{2} \int E^2 d\tau \qquad (2.45)$$

In 2.42, the energy is stored in the interaction between charges and it does not contain self-energy. In 2.45, the energy is stored in the field and it contains self-energy, so it is always positive.

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Consider two point charges at $\vec{r}_1$ and $\vec{r}_2$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_o} \frac{q_1}{|\vec{r} - \vec{r}_1|^3} (\vec{r} - \vec{r}_1) + \frac{1}{4\pi\epsilon_o} \frac{q_2}{|\vec{r} - \vec{r}_2|^3} (\vec{r} - \vec{r}_2)$$

E^2 contains 3 terms and eq. 2.45 can be written as

$$W = \frac{1}{32\pi^2\epsilon_o} \left[\int \frac{q_1^2}{|\vec{r} - \vec{r}_1|^4} d\tau + \int \frac{q_2^2}{|\vec{r} - \vec{r}_2|^4} d\tau \right] + \frac{q_1 q_2}{16\pi^2\epsilon_o} \int \frac{(\vec{r} - \vec{r}_1) \cdot (\vec{r} - \vec{r}_2)}{|\vec{r} - \vec{r}_1|^3 \cdot |\vec{r} - \vec{r}_2|^3} d\tau$$

The third term above is the interaction term, we can show that

$$W_{Int} = \frac{1}{4\pi\epsilon_o} \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|}$$

The first two terms are self-energy.

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2. Energy can be viewed as stored in the charge interaction with potential, or it can be viewed as stored in the electric field.
3. Superposition theorem does not work for the energy because the energy term contains a quadratic term

$$W_1 = \frac{\epsilon_o}{2}\int E_1^2 d\tau \qquad W_2 = \frac{\epsilon_o}{2}\int E_2^2 d\tau$$

Let $\vec{E}_{total} = \vec{E}_1 + \vec{E}_2$

But $W_{total} \neq W_1 + W_2$

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Conductors

A conductor has unlimited free electrons moving around. It does not take any energy to move these electrons. So the charge distribution inside a conductor will be such that there is no E field at all.

1. $\vec{E} = 0$, *inside a conductor*,
2. ρ = 0, inside a conductor,
3. If there is any net charges, then they will reside on the surface,
4. V = constant through out a conductor,
5. $\vec{E}$ is perpendicular to the surface just outside a conductor.

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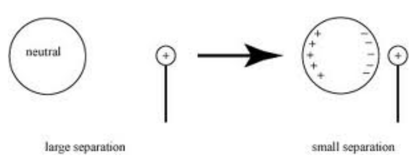
Induced Charges



A positive charge near a conductor will attract negative charges on the conductor to the near side and will repel positive charges to the far side. Because of this charge re-distribution, the conductor will be attractive to the charge. From action-reaction, we can say that a charge will be attractive to a conductor even if it is neutral.

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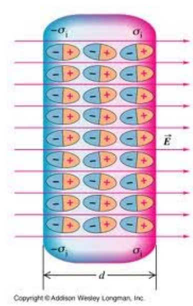
Even if we deal with dielectric which does not have free electron, we still observe the same phenomenon. This is due to the fact that even though there is no free charges, the atoms can be polarized.



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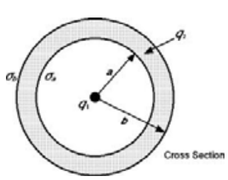
Example 9 and problem 2.36



To shield a charge inside a conductor, it is necessary to ground the conductor.

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Surface charge and the force on a conductor

From Gauss's Law

$$\oint \vec{E}\cdot\vec{da} = \frac{Q_{enc}}{\epsilon_o} = \frac{\sigma A}{\epsilon_o}$$

Since the field inside a conductor is zero, so

$$\vec{E}_{out} = \frac{\sigma}{\epsilon_o}\hat{n}$$

Since $\vec{E} = -\frac{\partial V}{\partial n}\hat{n}$, we can re-write above equation in terms of potential

$$\frac{\partial V}{\partial n} = -\frac{\sigma}{\epsilon_o} \quad \text{or} \quad \sigma = -\epsilon_o\frac{\partial V}{\partial n}$$

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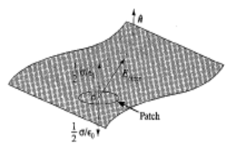
For a surface charge in an electric field, it will experience a force. The force on this charge density is given by

$$f = \sigma \vec{E}_{ave} = \frac{1}{2}\sigma(\vec{E}_{above} + \vec{E}_{below})$$

We can view the E field as

$$\vec{E} = \vec{E}_{patch} + \vec{E}_{other}$$



$$\vec{E}_{above} = \vec{E}_{other} + \frac{\sigma}{2\epsilon_o}\hat{n}$$

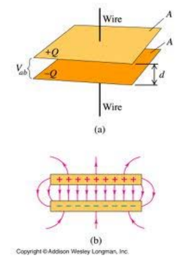
$$\vec{E}_{below} = \vec{E}_{other} - \frac{\sigma}{2\epsilon_o}\hat{n}$$

$$\vec{E} = \vec{E}_{other} = \vec{E}_{average}$$

Capacitor

Suppose we have two parallel conducting plates and we put +Q on one plate and –Q on the other plate as shown

As we can see that there is a potential difference between the two plates and there is an electric field between the two plates. We are interested in finding the relationship between the E field, the potential difference, and the amount of charge on the plate.



1. The potential difference is proportional to the $\vec{E}$ between the plates.

$$V = -\int_1^2 \vec{E} \cdot \vec{dl}$$

2. From the Gauss's law, we can see that the charge is proportional to the electric field between the plates.

$$\mathrm{E} = \frac{Q}{A\epsilon_o}$$

3. The amount of charge on the plate is also proportional to the potential difference. We define a quantity called capacitance as the ratio of charge to the electric potential difference. The capacitance is the amount of charge a "device" can hold per unit voltage.

$$C = \frac{Q}{V}$$

This is a very general expression can be used in any geometry.

For a parallel plate capacitor, the capacitance is given by

$$C = \frac{A\epsilon_o}{d}$$

This is a very specific expression, used for parallel plate capacitor only.

Example 11

For a parallel plate capacitor, the E field inside is given by

$$\vec{E} = \frac{\sigma}{\epsilon_o}\hat{n} = \frac{Q}{A\epsilon_o}\hat{n}$$

The potential difference between the two plates is

$$V = E \cdot d = \frac{Qd}{A\epsilon_o} \Rightarrow Q = \frac{A\epsilon_o}{d} \cdot V$$

$$\Rightarrow C = \frac{A\epsilon_o}{d}$$

To charge up a capacitor, we need to do "work"

$$W = \int dW = \int V dq = \int_0^Q \frac{q}{C} dq = \frac{1}{2}\frac{Q^2}{C}$$

Let Q=CV,

$$W = \frac{1}{2}CV^2$$

From eq. 2.43

$$W = \frac{1}{2}\int \rho V d\tau = \frac{1}{2}QV = \frac{1}{2}CV^2$$