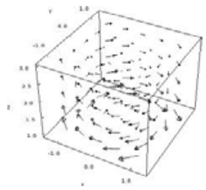
Chapter 1 Vector Analysis



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1. Vector Algebra

- Addition of two vectors

$\vec{A}+\vec{B}=\vec{B}+\vec{A}$ Communicative

$\vec{A}+(\vec{B}+\vec{C})=(\vec{A}+\vec{B})+\vec{C}$ Associative

$\vec{A}-\vec{B}=\vec{A}+(-\vec{B})$ Definition

- Multiplication by a scalar

$a(\vec{A}+\vec{B})=a\vec{A}+a\vec{B}$ Distribution

- Dot product of two vectors

$\vec{A}\cdot\vec{B}=AB\cos\theta, \quad \theta \text{ is angle between } \vec{A}\ \&\ \vec{B}$

$\vec{A}\cdot(\vec{B}+\vec{C})=\vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}$

$\vec{A}\cdot\vec{B}=\vec{B}\cdot\vec{A}$

- Cross-product of two vectors

$$\vec{A}\times\vec{B}=AB\sin\theta\hat{n}$$
where $\vec{n}\perp\vec{A}, \quad and \quad \hat{n}\perp\vec{B}$

In a strict sense, if $\vec{A}$ and $\vec{B}$ are vectors, the cross product of two vectors is a pseudo-vector.

A vector is defined as a mathematical quantity which transform like a position vector:

$$\vec{r}=x\hat{\imath}+y\hat{\jmath}+z\hat{k}$$

Cross-product follows distributive rule but not the commutative rule.

$$\vec{A}\times(\vec{B}+\vec{C})=\vec{A}\times\vec{B}+\vec{A}\times\vec{C}$$

Distribution rule

But

$$\vec{A}\times\vec{B}=-\vec{B}\times\vec{A}$$

so

$$\vec{A}\times\vec{B}\neq\vec{B}\times\vec{A}$$

Vector components

Even though vector operations is independent of the choice of coordinate system, it is often easier to set up Cartesian coordinates and work with the components of a vector.

$$\vec{A}=A_x\hat{x}+A_y\hat{y}+A_z\hat{z}$$

where $\hat{x}$, $\hat{y}$, and $\hat{z}$ are unit vectors which are perpendicular to each other. A_x, A_y, and A_z are components of $\vec{A}$ in the x-, y- and z-direction.

$$\vec{A}+\vec{B}=(A_x+B_x)\hat{x}+(A_y+B_y)\hat{y}+(A_z+B_z)\hat{z}$$

$$\vec{A}\cdot\vec{B}=A_xB_x+A_yB_y+A_zB_z$$

$$\vec{A}\times\vec{B}=(A_x\hat{x}+A_y\hat{y}+A_z\hat{z})\times(B_x\hat{x}+B_y\hat{y}+B_z\hat{z})$$
$$=(A_yB_z-A_zB_y)\hat{x}+(A_zB_x-A_xB_z)\hat{y}+(A_xB_y-A_yB_x)\hat{z}$$

$$=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}$$

Vector triple products

$$\vec{A}\cdot(\vec{B}\times\vec{C})=\vec{B}\cdot(\vec{C}\times\vec{A})=\vec{C}\cdot(\vec{A}\times\vec{B})$$

$$\vec{A}\cdot(\vec{B}\times\vec{C})=\begin{vmatrix}A_x & A_y & A_z\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix}$$

$$= - \begin{vmatrix}B_x & B_y & B_z\\ A_x & A_y & A_z\\ C_x & C_y & C_z\end{vmatrix}$$

$$\vec{A}\cdot(\vec{B}\times\vec{C})=(\vec{A}\times\vec{B})\cdot\vec{C}$$

$$\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B})$$

$$(\vec{A}\times\vec{B})\times\vec{C}=-\vec{A}(\vec{B}\cdot\vec{C})+\vec{B}(\vec{A}\cdot\vec{C})$$

All these vector products can be verified using the vector component method. It is usually a tedious process, but not a difficult process.

Position, displacement, and separation vectors

The location of a point (x, y, z) in Cartesian coordinate as shown below can be defined as a vector from (0,0,0) to (x, y, z) is given by

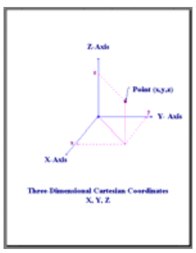
$$\vec{r}=x\hat{\imath}+y\hat{\jmath}+z\hat{k}$$

where

$$|\vec{r}|=r=\sqrt{x^2+y^2+z^2}$$

and

$$\hat{r}=\frac{\vec{r}}{r}=\frac{x\hat{\imath}+y\hat{\jmath}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}$$



Three Dimensional Cartesian Coordinates X, Y, Z

Infinitesimal displacement vector is given by

$$\widehat{d\ell}=dx\hat{x}+dy\hat{y}+dz\hat{z}$$

In general, when a charge is not at the origin, say at (x', y', z'), to find the $\vec{E}$ field produced by this charge at another position, (x, y, z), we use the following, which is called a separation vector,

$$\vec{\mathscr{r}}=\vec{r}-\vec{r'}$$

$$\vec{\mathscr{r}}=(x-x')\hat{x}+(y-y')\hat{y}+(z-z')\hat{z}$$

$$|\vec{\mathscr{r}}|=\mathscr{r}=\left|\vec{r}-\vec{r'}\right|$$

$$\hat{\mathscr{r}}=\frac{\vec{r}-\vec{r'}}{\left|\vec{r}-\vec{r'}\right|}$$

Vector transformation

So far we have used two different ways to describe vector; (a) geometry approach---vector as an arrow, (b) algebra approach--- vector as components of Cartesian coordinates. However both approaches are not very satisfactory and are rather naïve.

Here we follow the approach of a mathematician and define a vector as a set of three components that transforms in the same manner as a displacement when we change the coordinates. As always, the displacement vector is the model for the behavior of all vectors.

A more rigorous definition of vector starts with the concept that the space is (a) isotopic, so no preferred direction, (b) homogeneous, so no preferred location. A physical quantity such as displacement or force, should be independent of the coordinate system we choose.

Let $\vec{A}=\overline{OP}$

$$\vec{A}=A_y\hat{y}+A_z\hat{z}$$
$$\vec{A'}=A'_y\hat{y}'+A'_z\hat{z}'$$

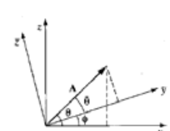


FIGURE 1.15

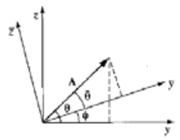


FIGURE 1.15

$$A'_y = A \cdot cos(\theta - \varphi)$$
$$= A \cdot (cos\theta \cdot cos\varphi + sin\theta \cdot sin\varphi)$$
$$= A_y cos\varphi + A_z sin\varphi$$

$$A'_z = A \cdot sin(\theta - \varphi)$$
$$= A \cdot (sin\theta \cdot cos\varphi - cos\theta \cdot sin\varphi)$$
$$= -A_y \cdot sin\varphi + A_z \cdot cos\varphi$$

The relationship between A'_y, A'_z and A_y, A_z can be expressed below:

$$A'_y = cos\varphi A_y + sin\varphi A_z$$
$$A'_z = -sin\varphi A_y + cos\varphi A_z$$

The above equations can be combined into one matrix notation;

$$\begin{pmatrix} A'_y \\ A'_z \end{pmatrix} = \begin{pmatrix} cos\varphi & sin\varphi \\ -sin\varphi & cos\varphi \end{pmatrix} \begin{pmatrix} A_y \\ A_z \end{pmatrix}$$

This is a rotation about the x-axis. For rotation about an arbitrary axis, the transformation law has the form:

$$\begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix} = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

The rotation matrix above describes the rotation transformation of a vector from one coordinate to another coordinate, it can be written more compactly as

$$A'_i = \sum_{j=1}^{3} R_{ij} A_j$$

This is how a displacement vector transformed. In general we define a vector as any set of three components that transform in the same manner as a displacement vector when we rotate the coordinates.

Similar idea can be extended to tensor; namely for a second rank tensor in 3D, the rotation through an arbitrary angle can be expressed as:

$$T'_{ij} = \sum_{k=1}^{3} \sum_{l=1}^{3} R_{ik} R_{jl} T_{kl}$$

Differential vector calculus

In 1D, the infinitesimal change of a function f(x), df is given by

$$df = \left(\frac{df(x)}{dx}\right) dx = \left(\frac{\partial f(x)}{\partial x}\right) dx$$

Where the derivative $^{df(x)}/_{dx}$ is the same as the partial derivative $^{\partial f(x)}/_{\partial x}$. The derivative is the slope of the function f(x).

For a function with two variables, f (x, y), the infinitesimal change of the function, *df*, is given by:

$$df \equiv \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy$$

Note that when we take partial derivative with respect to x, we need to hold the other variable y as a constant. This concept can extent to n-dimension.

In 3D with 3 variables, we have

$$df \equiv \left(\frac{\partial f}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y}\right) dy + \left(\frac{\partial f}{\partial z}\right) dz$$

This expression looks a lot like a vector dot product!

$$df \equiv \left(\frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}\right) \cdot (dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k})$$

$$df \equiv \nabla f \cdot \overrightarrow{dl}$$

$$\nabla f = \left(\frac{\partial f}{\partial x}\right)\hat{\imath} + \left(\frac{\partial f}{\partial y}\right)\hat{\jmath} + \left(\frac{\partial f}{\partial z}\right)\hat{k}$$

∇f is the gradient of the function f (x,y,z). ∇f is a vector field and pointing at the direction of maximum slope.

In general, the function f(x, y, z) is an arbitrary function, however, ∇f has some special properties it is not arbitrary any more.

Problem 1.11(b)

Find the gradient of the function, $f(x,y,z)=x^2y^3z^4$

$$\nabla f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$
$$= 2xy^3z^4\hat{x} + 3x^2y^2z^4\hat{y} + 4x^2y^3z^3\hat{z}$$

Example 1.3

Find the gradient of $r = \sqrt{x^2+y^2+z^2}$.

$$\nabla r = \frac{\partial r}{\partial x}\hat{x} + \frac{\partial r}{\partial y}\hat{y} + \frac{\partial r}{\partial z}\hat{z}$$
$$= \frac{x\hat{x}+y\hat{y}+z\hat{z}}{\sqrt{x^2+y^2+z^2}} = \frac{\vec{r}}{r} = \hat{r}$$

The operator "del" ∇

∇ is a vector, and it is also an operator.

$$\nabla \equiv \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

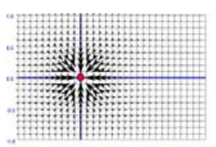
As an operator, it always operates on a function.
As a vector, it can operates on a scalar, or a vector.

$\nabla f(x,y,z)$ -------- Gradient of f(x,y,z)
$\nabla \cdot \vec{V}(x,y,z)$ -------- Divergent of $\vec{V}$
$\nabla \times \vec{V}(x,y,z)$ ------- Curl of $\vec{V}$

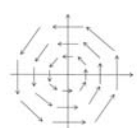
Divergence of a vector field

$$\nabla \cdot \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

The divergence of a vector field is a measure of how the vector field "spread out from a given point".



$$\vec{V} = \frac{-x\hat{\imath}}{\sqrt[3]{x^2+y^2}} + \frac{-y\hat{\jmath}}{\sqrt[3]{x^2+y^2}}$$

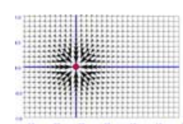


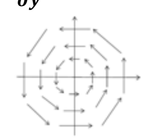
$$\vec{V} = -y\hat{\imath} + x\hat{\jmath}$$

$$\nabla \cdot \left[\frac{-x\hat{\imath}}{\sqrt[3]{x^2+y^2}} + \frac{-y\hat{\jmath}}{\sqrt[3]{x^2+y^2}}\right] = \frac{\partial}{\partial x}\left[\frac{-x}{\sqrt[3]{x^2+y^2}}\right] + \frac{\partial}{\partial y}\left[\frac{-y}{\sqrt[3]{x^2+y^2}}\right]$$

$$= \left[\frac{-1}{\sqrt[3]{x^2+y^2}} + \frac{3x^2}{\sqrt[5]{x^2+y^2}}\right] + \left[\frac{-1}{\sqrt[3]{x^2+y^2}} + \frac{3y^2}{\sqrt[5]{x^2+y^2}}\right]$$

$$\nabla \cdot [-y\hat{\imath} + x\hat{\jmath}] = \frac{\partial}{\partial x}[-y] + \frac{\partial}{\partial y}[-x] = 0$$





The Curl

The Curl of a vector field is defined as

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$$= \hat{\imath}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \hat{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

The Curl measures how a vector field "curl" around a point. Curl is a vector operator, but in general, <u>$\nabla \times \vec{V}$ not necessarily perpendicular to $\vec{V}$</u>

Example (This is not the same as example on page 21)

$\vec{V} = -z\hat{\imath} + x\hat{\jmath}$, $\nabla \cdot \vec{V} = 0$, and $\nabla \times \vec{V}$ is calculated below.

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z & x & 0 \end{vmatrix} = -\hat{\jmath} + \hat{k}$$

$$(\nabla \times \vec{V}) \cdot \vec{V} = -x$$

$(\nabla \times \vec{V})$ *is not* $\perp$ to $\vec{V}$

Vector Product Rules

Page 20-21, there are many rules. It is useful to work out some of the expressions in details yourself at home.

Second derivatives

(a) Divergence of a gradient

$$\nabla\cdot(\nabla f)=\left[\hat{\imath}\frac{\partial}{\partial x}+\hat{\jmath}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right]\cdot\left[\frac{\partial f}{\partial x}\hat{\imath}+\frac{\partial f}{\partial y}\hat{\jmath}+\frac{\partial f}{\partial z}\hat{k}\right]$$

$$=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}=\nabla^2 f$$

This is called the Laplacian of *f(x,y,z)*. This is a very common operator, which we encounter in mathematical physics quite often.

(b) Curl of a gradient **(Always equals to zero)**

$$\nabla\times\nabla f=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z}\end{vmatrix}$$

$$=\left(\frac{\partial^2 f}{\partial y\partial z}-\frac{\partial^2 f}{\partial z\partial y}\right)\hat{\imath}+\left(\frac{\partial^2 f}{\partial z\partial x}-\frac{\partial^2 f}{\partial x\partial z}\right)\hat{\jmath}+\left(\frac{\partial^2 f}{\partial x\partial y}-\frac{\partial^2 f}{\partial y\partial x}\right)\hat{k}$$

$$=0$$

(c) Gradient of a divergence **(Seldom used)**

$$\nabla(\nabla\cdot\vec{A})=\left[\hat{\imath}\frac{\partial}{\partial x}+\hat{\jmath}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right]\left[\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right]$$

$$=\left(\frac{\partial^2 A_x}{\partial x^2}+\frac{\partial^2 A_y}{\partial x\partial y}+\frac{\partial^2 A_z}{\partial x\partial z}\right)\hat{\imath}+\left(\frac{\partial^2 A_x}{\partial y\partial x}+\frac{\partial^2 A_y}{\partial y^2}+\frac{\partial^2 A_z}{\partial y\partial z}\right)\hat{\jmath}$$

$$+\left(\frac{\partial^2 A_x}{\partial z\partial x}+\frac{\partial^2 A_y}{\partial z\partial y}+\frac{\partial^2 A_z}{\partial z^2}\right)\hat{k}$$

Note: $\nabla(\nabla\cdot\vec{A})\neq(\nabla\cdot\nabla)\vec{A}$

(d) The divergence of a Curl **(Always = 0)**

$$\nabla\cdot(\nabla\times\vec{A})=\left(\hat{\imath}\frac{\partial}{\partial x}+\hat{\jmath}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot$$

$$\left[\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\hat{\imath}+\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\hat{\jmath}+\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\hat{k}\right]$$

$$=\left[\frac{\partial^2 A_z}{\partial x\partial y}-\frac{\partial^2 A_y}{\partial x\partial z}\right]+\left[\frac{\partial^2 A_x}{\partial y\partial z}-\frac{\partial^2 A_z}{\partial y\partial x}\right]+\left[\frac{\partial^2 A_y}{\partial z\partial x}-\frac{\partial^2 A_x}{\partial z\partial y}\right]=0$$

(e) The Curl of a Curl

$$\nabla\times(\nabla\times\vec{A})=\nabla(\nabla\cdot\vec{A})-\nabla^2\vec{A}$$

$$\nabla\times(\nabla\times\vec{A})=\left[\left(\frac{\partial^2 A_y}{\partial x\partial y}-\frac{\partial^2 A_x}{\partial y^2}\right)-\left(\frac{\partial^2 A_x}{\partial z^2}-\frac{\partial^2 A_z}{\partial x\partial z}\right)\right]\hat{\imath}$$

$$+\left[\left(\frac{\partial^2 A_z}{\partial y\partial z}-\frac{\partial^2 A_y}{\partial z^2}\right)-\left(\frac{\partial^2 A_y}{\partial x^2}-\frac{\partial^2 A_x}{\partial x\partial y}\right)\right]\hat{\jmath}$$

$$+\left[\left(\frac{\partial^2 A_x}{\partial x\partial z}-\frac{\partial^2 A_z}{\partial x^2}\right)-\left(\frac{\partial^2 A_z}{\partial y^2}-\frac{\partial^2 A_y}{\partial y\partial z}\right)\right]\hat{k}$$

1-3 Integral calculus

Let's start with a 1D function, F(x), which is the derivative of another function, f(x)

$$F(x) = \frac{df(x)}{dx}$$

Then

$$\int_a^b F(x)dx = \int_a^b \frac{df(x)}{dx} dx = \int_a^b df(x) = f(b) - f(a).$$

The result of the integral only depends on the end point of f(x), because in 1D, the boundary of an arbitrary curve is the end point.

In general, when we are in a 3D space, the line integral of an arbitrary function $\vec{V}(x, y, z)$

$$\int_a^b \vec{V}(x, y, z) \cdot \vec{dl}$$

will depend on the path taken.

However, if we have a specific function, namely

$$\vec{F}(x, y, z) = \nabla f(x, y, z)$$

Then the line integral of a gradient is independent of the path taken.

$$\int_a^b (\nabla f) \cdot \vec{dl} = f(b) - f(a)$$

The fundamental theorem of Gradients

This is called the "Fundamental theorem for gradients", and

$$\oint (\nabla f) \cdot \vec{dl} = 0$$

In physics, we said that if we can associate a potential to a field (force), then the field (force) is conservative. Namely the work done by the field does not depend on the path.

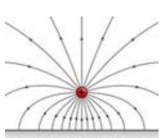
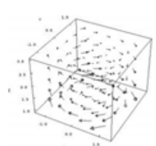
The fundamental theorem of divergence.

The divergence theorem states that:

$$\int_V (\nabla \cdot \vec{V}) d\tau = \oint_S \vec{V} \cdot \vec{da}$$

The divergence theorem is very similar to the gradient theorem, namely, the integral (volume) of a derivative (divergence) of a function is equal to the function evaluated at the boundary (surface).

It is also called Gauss's theorem or Green's theorem.

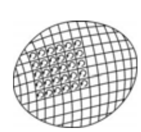
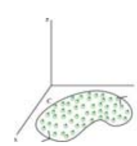



The volume integral of the divergence of a vector field over the whole volume is equal to the value of the vector field evaluated at the surface which bounds the volume.

Physically this divergence theorem translates into conservation law again. For example, for a non-compressible fluid, $\nabla \cdot \vec{V}$ is a measure of the source strength, and $\vec{V} \cdot \vec{da}$ is a measure of the flux.

The fundamental theorem of Curls

$$\int_S (\nabla \times \vec{V}) \cdot \vec{da} = \oint_{Line} \vec{V} \cdot \vec{dl}$$ Stoke's theorem

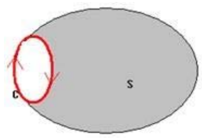



If the surface is a closed surface,

$$\oint_S (\nabla \times \vec{V}) \cdot \vec{da} = 0$$

The Stoke's theorem states that the surface integral of a Curl is only depended on the boundary condition, namely the line integral of the "edge" of the surface and does not depend on the surface we choose, as long as the boundaries are the same.

For example:



Relations among the fundamental theorems

(1) Gradient Theorem:

$$\int_{Line} (\nabla f)\cdot \vec{dl} = f(b) - f(a)$$

(2) Divergence theorem:

$$\int_{\tau} (\nabla \cdot \vec{V}) d\tau = \oint_{A} \vec{V}\cdot \vec{da}$$

(3) Stoke's theorem:

$$\int_{S} (\nabla \times \vec{V})\cdot \vec{da} = \oint_{Line} \vec{V}\cdot \vec{dl}$$

These are boundary value problems, relating the integral of a "derivative" of the function to the evaluation of the function at its boundary.

From the previous page, if we combine equations (1) and (3), (namely let $\vec{V} = \nabla f$)

$$\int [\nabla \times (\nabla f)]\cdot \vec{da} = 0 \quad \text{For any surface}$$

$$\nabla \times (\nabla f) = 0$$

If we combine (2) and (3),

$$\int_{V} \nabla \cdot (\nabla \times \vec{V}) d\tau = \oint (\nabla \times \vec{V})\cdot \vec{da} = 0$$

$$\nabla \cdot (\nabla \times \vec{V}) = 0$$

Note: In general, we can start with an arbitrary vector field, $\vec{V}(x,y,z)$, or an arbitrary function, f(x,y,z).

However, ∇f or $\nabla \times \vec{V}$ are not arbitrary function any more, namely;

∇f **is a Curl-free vector field,**

and

$\nabla \times \vec{A}$ **is a divergence-free field.**

Integration by parts

From product rule for the derivatives

$$\frac{d}{dx}(fg) = f\left(\frac{dg}{dx}\right) + g\left(\frac{df}{dx}\right)$$

We can see that by integrating both sides, we end up with

$$\int_a^b d(fg) = \int_a^b f\left(\frac{dg}{dx}\right)dx + \int_a^b g\left(\frac{df}{dx}\right)dx$$

This can be extended to vector product rule such as

$$\nabla \cdot (f\vec{A}) = f(\nabla \cdot A) + \vec{A}\cdot(\nabla f)$$

Integrating over a volume, we end up with

$$\int \nabla \cdot (f\vec{A}) d\tau = \int f(\nabla \cdot \vec{A}) d\tau + \int \vec{A}\cdot(\nabla f) d\tau$$

Re-arrange and applying the divergent theorem

$$\oint f\vec{A}\cdot \vec{da} = \int f(\nabla \cdot \vec{A}) d\tau + \int \vec{A}\cdot(\nabla f) d\tau$$

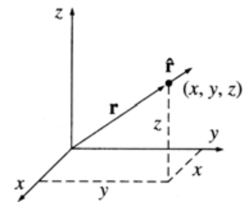
Curvilinear Coordinates

(a)Cartesian Coordinates

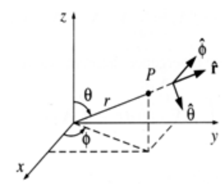
(b)Spherical Coordinates

(c)Cylindrical coordinates

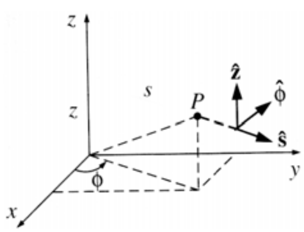
These coordinates can be shown on the next two pages



Cartesian Coordinates
(x, y, z)



Spherical Coordinates
(r, θ, φ)



Cylindrical Coordinates
(s, φ, z)

They are all orthogonal curvilinear coordinates. The three directions of each coordinate satisfies the cyclic relationship.

Cartesian coordinates

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient: $\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

Curl: $\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$

Laplacian: $\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

Spherical Coordinates

Spherical: $d\vec{l} = dr\hat{r} + rd\theta\hat{\theta} + rsin\theta d\varphi\hat{\varphi}$, $\quad d\tau = r^2 sin\theta dr d\theta d\varphi$

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}}.$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}.$$

Curl:

$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}.$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}.$$

Cylindrical Coordinates

Cylindrical: $d\vec{l} = ds\hat{s} + sd\varphi\hat{\varphi} + dz\hat{z}$, $\quad d\tau = sdsd\varphi dz$

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}.$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}.$$

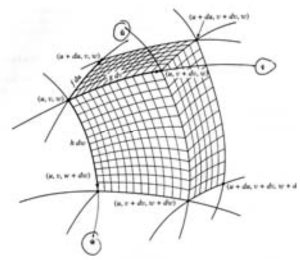
Curl:

$$\nabla \times \mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}.$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}.$$

These complicated formulism can be understood through orthogonal curvilinear coordinates.



Gradient in an arbitrary orthogonal curvilinear system

Assume we have a function t(u, v, w), where u, v, w are orthogonal, the change of t from t(u, v, w) to t(u+du, v+dv, w+dw) is given by

$$dt = \frac{\partial t}{\partial u}du + \frac{\partial t}{\partial v}dv + \frac{\partial t}{\partial w}dw$$

This can be written as a dot product

$$dt = \nabla t \cdot d\vec{l} = (\nabla t)_u f du + (\nabla t)_v g dv + (\nabla t)_w h dw$$

where

$$d\vec{l} = f du \hat{u} + g dv \hat{v} + h dw \hat{w}$$

and

$$(\nabla t)_u \equiv \frac{1}{f}\frac{\partial t}{\partial u}, \quad (\nabla t)_v \equiv \frac{1}{g}\frac{\partial t}{\partial v}, \quad (\nabla t)_w \equiv \frac{1}{h}\frac{\partial t}{\partial w}$$

From the last equation from page 50, we can see that

$$\nabla t \equiv \frac{1}{f}\frac{\partial t}{\partial u}\hat{u} + \frac{1}{g}\frac{\partial t}{\partial v}\hat{v} + \frac{1}{h}\frac{\partial t}{\partial w}\hat{w}$$

For the most common coordinates

System	u	v	w	f	g	h
Cartesian	x	y	z	1	1	1
Spherical	r	θ	ϕ	1	r	$r\sin\theta$
Cylindrical	s	ϕ	z	1	s	1

Similarly

$$\nabla \cdot \mathbf{A} \equiv \frac{1}{fgh}\left[\frac{\partial}{\partial u}(ghA_u) + \frac{\partial}{\partial v}(fhA_v) + \frac{\partial}{\partial w}(fgA_w)\right],$$

$$\nabla \times \mathbf{A} \equiv \frac{1}{gh}\left[\frac{\partial}{\partial v}(hA_w) - \frac{\partial}{\partial w}(gA_v)\right]\hat{\mathbf{u}} + \frac{1}{fh}\left[\frac{\partial}{\partial w}(fA_u) - \frac{\partial}{\partial u}(hA_w)\right]\hat{\mathbf{v}} + \frac{1}{fg}\left[\frac{\partial}{\partial u}(gA_v) - \frac{\partial}{\partial v}(fA_u)\right]\hat{\mathbf{w}},$$

$$\nabla^2 t \equiv \frac{1}{fgh}\left[\frac{\partial}{\partial u}\left(\frac{gh}{f}\frac{\partial t}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{fh}{g}\frac{\partial t}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{fg}{h}\frac{\partial t}{\partial w}\right)\right].$$

The Dirac Delta Function

The divergence of $\vec{V} = \hat{r}/r^2$

This Coulomb field has a $1/r^2$ dependence. The field is shown above and it spreads out from the origin. So intuitively we can see that $\nabla \cdot (\hat{r}/_{r^2})$ is non-zero. However, if we take the divergence, we will find that

$$\nabla \cdot (\hat{r}/_{r^2}) = \frac{1}{r^2 sin\theta}\frac{\partial}{\partial r}\left(r^2 sin\theta \cdot \frac{1}{r^2}\right) = 0 \quad ????$$

If we look closer we will find that

For $r \neq 0$, $\nabla \cdot \vec{V} = 0$
For $r = 0$, $\nabla \cdot \vec{V} = \infty$

From the divergence theorem, we know that:

$$\int_\tau (\nabla \cdot \vec{V}) \cdot d\tau = \oint \vec{V} \cdot \vec{da}$$

$$\vec{V} = \frac{\hat{r}}{r^2} \quad \text{and} \quad \vec{da} = Rd\theta \cdot Rsin\theta d\varphi \hat{r}$$

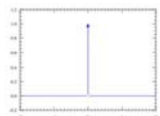
$$RHS \quad \oint \vec{V} \cdot \vec{da} = \frac{1}{R^2}R^2 \int sin\theta \cdot d\theta \int d\varphi = 4\pi$$

$$LHS \quad \int_V (\nabla \cdot \vec{V}) d\tau = \int_V [4\pi . \delta^3(r)] d\tau = 4\pi \int_V \delta^3(\vec{r}) d\tau$$

$$\nabla \cdot \frac{\hat{r}}{r^2} = 4\pi\delta^3(\vec{r}) \qquad \int_V \delta^3(\vec{r}) d\tau = 1$$

The Concept of delta function

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases} \qquad \int_{-\infty}^{\infty} \delta(x)dx = 1$$



$$\int_{-\infty}^{\infty} \delta(x)f(x)dx = f(0), \qquad \int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$$

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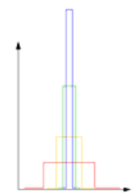
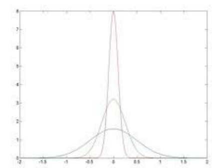
The Dirac delta function is invented by Paul Dirac in the early 1930. In a strict mathematical sense, $\delta(x)$ is not a function at all because it has a singularity at the origin.

There are many ways to construct a delta function.





(1) $\delta(x) = \lim_{n\to\infty} R_n(x)$ (2) $\delta(x) = \lim_{n\to\infty} T_n(x)$

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Different forms of delta function

3. $\delta(x) = \lim_{\lambda\to\infty} \frac{Sin\lambda x}{\pi x}$

4. $\delta(x) = \lim_{\lambda\to\infty} \frac{\lambda}{\pi(\lambda^2 x^2+1)}$

5. $\delta(x) = \lim_{\epsilon\to 0} \frac{\varepsilon}{\pi(x^2+\epsilon^2)}$

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6. $\delta(x) = \frac{1}{\sqrt{\pi}} \lim_{\epsilon\to 0} \frac{1}{\sqrt{\epsilon}} e^{\left(-\frac{x^2}{\epsilon}\right)}$

7. $\delta(x) = \frac{1}{2}\frac{d^2}{dx^2}|x|$

8. $\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iux}du$

We can use $\delta(x)$ to define a step function, S(x) as follow

$$S(x) = \int_{-\infty}^{x} \delta(x')dx' = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

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Properties of $\delta(x)$

1. $\int_{-\infty}^{\infty} f(x)\cdot\delta(x)dx = f(0)$

2. $\int_{-\infty}^{\infty} \delta(x)dx = 1$

3. $f(x)\delta(x) = f(0)\delta(x)$

4. $\delta(kx) = \frac{1}{|k|}\delta(x)$

Proof

$$\int_{-\infty}^{\infty} f(x)\delta(kx)dx = \pm\int_{-\infty}^{\infty} f(y/k)\delta(y)\frac{dy}{k}$$

$$= \pm\frac{1}{k}f(0) = \frac{1}{|k|}f(0) = \int_{-\infty}^{\infty} f(x)\frac{\delta(x)}{|k|}dx$$

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5. $\delta(ax-b) = \delta\left(a\left(x-\frac{b}{a}\right)\right) = \frac{1}{|a|}\delta(x-\frac{b}{a})$

6. $\int_{-\infty}^{\infty} \delta(x-a)\delta(x-b)\,dx = \delta(a-b)$

7. $\int f(x)\delta'(x)dx = \int [f(x)\delta(x)]'dx - \int f'(x)\delta(x)dx$

$= \frac{d}{dx}[\int f(x)\delta(x)dx] - f'(0) = -f'(0)$

8. $\int_{-\infty}^{\infty} g(x)\delta(f(x))dx = \sum_i \frac{g(x)}{\left|\frac{df}{dx}\right|}|_{f(x_i)}$

9. $\delta(x^2-a^2) = \frac{1}{2a}[\delta(x-a)+\delta(x+a)]$

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Dirac delta function is closely related to Fourier analysis. Consider a "complete" orthonormal set of functions $\varphi_n(x)$,

$$\int \varphi_n(x)\varphi_m^*(x)dx = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

"Complete" means that any arbitrary function f(x) can be expressed in terms of $\varphi_n(x)$

$$f(x) = \sum_n a_n\varphi_n(x) \qquad (1)$$

the coefficients a_n is

$$a_n = \int_{-\infty}^{\infty} f(x')\varphi_n^*(x')dx'$$

Now if we substitute a_n back into eq.(1) on page 61

$$f(x) = \sum_n \left[\int_{-\infty}^{\infty} f(x')\varphi_n^*(x')dx' \right] \cdot \varphi_n(x)$$

$$= \int_{-\infty}^{\infty} f(x')[\textstyle\sum_n \varphi_n^*(x')\varphi_n(x)]dx'$$

$$\delta(x - x') = \sum_n \varphi_n^*(x')\varphi_n(x)$$

If φ_n is a continuous function

$$\delta(x - x') = \int_{-\infty}^{\infty} \varphi_k^*(x')\varphi_k(x)dk$$

We can see that any orthonormal set of functions can be used to derive the delta function.

3D delta function

$$\delta^3(\vec{r}) = \frac{1}{(2\pi)^3}\int e^{i\vec{k}\cdot\vec{r}}d^3k$$

Cartesian $\quad \delta^3(\vec{r}) = \delta(x)\delta(y)\delta(z)$

Spherical $\quad \delta^3(\vec{r}) = \frac{1}{r^2 sin\theta}\delta(r)\delta(\theta)\delta(\varphi)$

$$\int \delta^3(\vec{r})d\tau = \int \delta^3(\vec{r})r^2 sin\theta drd\theta d\varphi$$
$$= \int \delta(r)dr \cdot \int \delta(\theta)d\theta \cdot \int \delta(\varphi)d\varphi = 1$$

Cylindrical $\quad \delta^3(\vec{r}) = \frac{1}{r}\delta(r)\delta(\varphi)\delta(z)$

$$\int \delta^3(\vec{r}) \cdot dr \cdot rd\varphi \cdot dz = \int \delta(r)dr \cdot \int \delta(\varphi)d\varphi \cdot \int \delta(z)dz = 1.$$

Now if we go back to the Coulomb field,

$$\nabla \cdot \frac{\hat{r}}{r^2} = 4\pi\delta^3(\vec{r}) \qquad \nabla\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}$$

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(\vec{r})$$

The above equations also can be applied to the separation Vector .

$$\nabla\left(\frac{1}{\boldsymbol{\mathscr{r}}}\right) = -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^2} \quad ; \quad \nabla^2\left(\frac{1}{\boldsymbol{\mathscr{r}}}\right) = -4\pi\delta^3(\vec{\boldsymbol{\mathscr{r}}})$$

The mathematical theory of Vector Fields

Here we want to prove that through the laws of electricity and magnetism and the boundary conditions, there always exists a unique vector field as the solution.

We want to find out, to what extent is a vector field determined by it divergence and curl? Namely if

$$\nabla \cdot \vec{F} = D \quad \text{and} \quad \nabla \times \vec{F} = \vec{C} \quad \text{are known}$$

Can we determine the vector field $\vec{F}$ uniquely?

(1) Does $\vec{F}$ exist?

(2) Is it unique?

If the source is known, can we determine the field?

First we will show that $\vec{F}$ does exist.

Let $\quad \vec{F} = -\nabla U + \nabla \times \vec{W}$

where

$$U(\vec{r}) = \frac{1}{4\pi}\int_V \frac{D(r')}{|\vec{r} - \vec{r'}|}d\tau'$$

$$\vec{W}(\vec{r}) = \frac{1}{4\pi}\int_V \frac{\vec{C}(\vec{r'})}{|\vec{r} - \vec{r'}|}d\tau'$$

Now take the divergence of $\vec{F}$, $\nabla \cdot \vec{F} = -\nabla^2 U$

$$\nabla \cdot \vec{F} = -\nabla^2 U = -\frac{1}{4\pi}\int D(r')\nabla^2\left(\frac{1}{r}\right)d\tau'$$

$$= \frac{1}{4\pi}\int D(r')\,4\pi\delta^3(\vec{r}-\vec{r'})d\tau' = D(\vec{r})$$

When we take the curl of $\vec{F}$, we have

$$\nabla \times \vec{F} = \nabla \times (\nabla \times \vec{W}) = -\nabla^2\vec{W} + \nabla(\underbrace{\cancel{\nabla \cdot \vec{W}}}_{0})$$

$$-\nabla^2\vec{W} = -\frac{1}{4\pi}\int \vec{C}(\vec{r'})\cdot\nabla^2\left(\frac{1}{r}\right)d\tau'$$

$$= \int \vec{C}(\vec{r'})\delta^3(\vec{r}-\vec{r'})d\tau' = \vec{C}(\vec{r})$$

So the function $\vec{F}$ exist !!!

The uniqueness of the vector field of a particular boundary condition is more difficult to prove.

If the derivative of the C function is finite everywhere, then the solution F is unique.

Scalar and vector potentials

Finally we will review briefly two specific vector fields that are commonly encountered in this course.

Curl-less field **A conservative field**

$\nabla \times \vec{F} = 0$ everywhere

$\Longrightarrow$ $\vec{F} = -\nabla U$

$\Longrightarrow$ $\int_a^b \vec{F}\cdot\vec{dl}$ independent of path

$\Longrightarrow$ $\oint \vec{F}\cdot\vec{dl} = 0$

Divergence-less field

$\nabla \cdot \vec{F} = 0$ everywhere

$\Longrightarrow$ $\vec{F} = \nabla \times \vec{W}$

$\Longrightarrow$ $\int_s \vec{F}\cdot\vec{da}$ independent of the surface, and only depends on the boundary of S.

$\Longrightarrow$ $\oint \vec{F}\cdot\vec{da} = 0$

Do Problem 1-55 in class.