Final Exam REAL

Name: SOLUTION

1	/15
2	/15
3	/15
4	/15
5	/15
6	/15
7	/15
8	/10
	115

1. Carnot cycle: Two identical bodies with temperature independent heat capacities C are initially at different temperatures T_H and T_C. A Carnot cycle is run between them (with infinitesimal steps) until they have been reduced to a common temperature T_F. [15 points]

a) Does the efficiency of this engine change over time? [5 points]

YES

b) In a Carnot cycle, the entropy change is zero per cycle as we saw in the previous exam. Find T_F in terms of T_H and T_C. The answer is not $(T_H+T_C)/2$. Consider the entropy change in the hot and cold reservoir: these have to be equal. Integrate from initial temperatures to find the solution. [5 points]

$$-\frac{dQ_H}{T_H} = \frac{dQ_C}{T_C}$$

$$-\frac{\cancel{C_0}\, dT_H}{T_H} = \frac{\cancel{C_0}\, dT_C}{T_C}$$

$$\int_{T_H}^{T_F} -\frac{\cancel{C_0}\, dT_H}{T_H} = \int_{T_C}^{T_F} \frac{\cancel{C_0}\, dT_C}{T_C}$$

$$-\ln\frac{T_F}{T_H} = \ln\frac{T_F}{T_C}$$

$$\frac{T_H}{T_F} = \frac{T_F}{T_C}$$

$$\boxed{T_F = \sqrt{T_H T_C}}$$

c) Find the total work done on the outside world in this process. Is it positive or negative? [5 points]

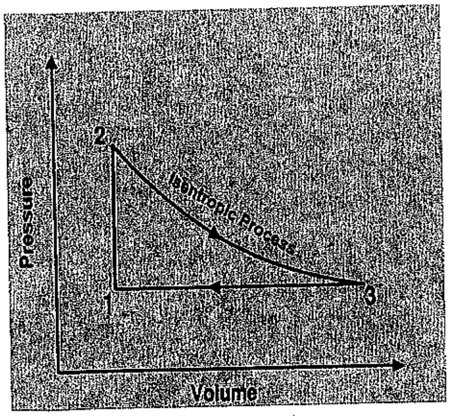
$$W = Q_{H\,TOTAL} - Q_{C\,TOTAL} = C_0(T_H - T_F) - C_0(T_F - T_C)$$

$$\boxed{W = C_0(T_H + T_C - 2T_F)}$$ 2 POINTS

$$\boxed{T_F = \sqrt{T_H T_C}}$$

$\boxed{\text{POSITIVE}}$ ⟵ GIVE 3 POINTS FOR THIS

2. Consider a Lenoir cycle as given by the figure below. We note that efficiency can be given by (Heat added)-(Heat rejected)/Heat added. Relevant temperatures are T_1, T_2, T_3. (Isentropic process = reversible adiabatic process). [15 points]



(a) Calculate T_3 in terms of T_2 [5 points]

$$ds = 0$$
$$dU = W$$
$$\frac{3}{2} Nk\,dT = -p\,dV = -\frac{NkT\,dV}{V}$$
$$\frac{3}{2}\frac{dT}{T} = -\frac{dV}{V}$$

$$\rightarrow \frac{3}{2}\ln\frac{T_3}{T_2} = -\ln\frac{V_3}{V_2}$$
$$\left(\frac{T_3}{T_2}\right)^{3/2} = \frac{V_2}{V_3}$$
$$\boxed{T_3 = T_2\left(\frac{V_2}{V_3}\right)^{2/3}}$$

(b) When is heat being added? [2.5 points]

$\boxed{1 \rightarrow 2}$

(c) When is heat being rejected? [2.5 points]

$\boxed{3 \rightarrow 1}$

(d) Calculate efficiency in terms of T_1, T_2, and T_3. as well as Nk (assume monoatomic gas is involved) [5 points] [Hint: C_v=3/2 Nk, C_p=5/2Nk, $dU+pdV = \Delta H = C_p\Delta T$]

Q_{ADDED} =

$$Q_{ADD} = C_v(T_2 - T_1) = \frac{3}{2}Nk(T_2 - T_1)$$

Q SUBTRACTED

$$dU = Q - p\,dV$$
$$dU + p\,dV = Q = \underline{\underline{\Delta H}}$$

$$|Q| = \frac{5Nk}{2}(T_3 - T_1)$$

$$\varepsilon = \frac{\frac{3}{2}Nk(T_2 - T_1) - \frac{5}{2}Nk(T_3 - T_1)}{\frac{3}{2}Nk(T_2 - T_1)}$$

$$\boxed{\varepsilon = 1 - \frac{5}{3}\frac{(T_3 - T_1)}{(T_2 - T_1)}}$$

3. Manipulation of thermodynamic quantities [15 points]

In a weakly interacting gas of Bose particles at low temperature the expansion coefficient α and the isothermal compressibility $\mathcal{K}_T$ are given by

$$\alpha \equiv \frac{1}{V}\frac{\partial V}{\partial T}\bigg|_P = \frac{5}{4}\frac{a}{c}T^{3/2}V^2 + \frac{3}{2}\frac{b}{c}T^2V^2$$

$$\mathcal{K}_T \equiv -\frac{1}{V}\frac{\partial V}{\partial P}\bigg|_T = \frac{1}{2c}V^2$$

where a, b and c are constants. It is known that the pressure goes to zero in the limit of large volume and low temperature. Find the equation of state $P(T, V)$.

Hint: You may use that (dP/dT) at constant volume $=\alpha/\kappa_T$

$P(T, V)$

$$dP = \underbrace{\left(\frac{\partial P}{\partial T}\right)_P}dT + \underbrace{\left(\frac{\partial P}{\partial V}\right)_T}dV$$

$$\frac{1}{\frac{\partial V}{\partial P}} = \frac{1}{-\frac{1}{2c}V^3} = -\frac{2c}{V^3}$$

$$\frac{\alpha}{\kappa_T} = \frac{5}{2}aT^{3/2} + 3bT^2 = \left(\frac{\partial P}{\partial T}\right)_P$$

$$dP = \left(\frac{5}{2}aT^{3/2} + 3bT^2\right)dT + \left(-\frac{2c}{V^3}\right)dV$$

$$P = \frac{5}{2}a\frac{2}{5}T^{5/2} + \frac{3bT^3}{3} + \frac{c}{V^2} + \text{CONST}$$

CONST 0

$$\boxed{P = aT^{5/2} + bT^3 + \frac{c}{V^2}}$$

4. Consider a simple harmonic oscillator with energy given by E = nħω, where ω is a constant given by sqrt (k/m). [15 points]

(a) Calculate its partition function. You may utilize this formula below to simplify your infinite sum. $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$ [3 points]

$$Z = \sum_{n=0}^{\infty} e^{-n\hbar\omega\beta} = \frac{1}{1 - e^{-\hbar\omega\beta}}$$

(b) What is its energy at some temperature T? [3 points]

$$\langle E \rangle = -\frac{\partial}{\partial\beta} \ln Z = -\frac{\partial}{\partial\beta}\left[-\ln\left(1 - e^{-\hbar\omega\beta}\right)\right]$$

$$\boxed{\langle E \rangle = \frac{\hbar\omega\, e^{-\hbar\omega\beta}}{1 - e^{-\hbar\omega\beta}}}$$

(c) What is the meaning of <n>? (Average n) It is not an occupation number. [3 points]

$\boxed{\frac{\langle E \rangle}{\hbar\omega}}$ OR $\boxed{\text{AVERAGE STATE \# OF S.H.O}}$

(d) Calculate heat capacity of this simple harmonic oscillator. [3 points]

$$\frac{\partial E}{\partial T} = \frac{\partial E}{\partial \beta}\frac{\partial \beta}{\partial T} = -\frac{\hbar\omega}{kT^2}\frac{\partial}{\partial\beta}\left(\frac{1}{e^{\hbar\omega\beta} - 1}\right)$$

$$= +\frac{\hbar\omega}{kT^2}\frac{\hbar\omega\, e^{\hbar\omega\beta}}{(e^{\hbar\omega\beta} - 1)^2} = \boxed{\frac{\hbar^2\omega^2\, e^{\hbar\omega\beta}}{kT^2(e^{\hbar\omega\beta} - 1)^2}}$$

(e) Calculate heat capacity at high temperature (kT>> ħω) [3 points]

AT HIGH $\beta \to 0$ $\quad e^{\hbar\omega\beta} = 1 + \hbar\omega\beta - 1 = \hbar\omega\beta$

$$C_V = \frac{\cancel{\hbar^2\omega^2}}{kT^2}\frac{1}{\cancel{\hbar^2\omega^2}\beta^2} = \boxed{k}$$

5. If a partition function for a single particle is given by Z, find out the partition function for N particles [15 points]

(a) provided that they are distinguishable particles [7.5 points]

$$Z_{TOTAL} = Z^N$$

(b) provided that they are indistinguishable particles [7.5 points]

$$Z_{TOTAL} = \frac{Z^N}{N!}$$

6. Consider a system composed of N spins in which energy levels are defined to be -μB, 0, μB and magnetic moment is for these levels are given by μ, 0, -μ.

(a) What is the partition function? [3 points]

$$Z = 1 + e^{-\mu B \beta} + e^{\mu B \beta}$$

(b) What is the probability of finding a spin with magnetic moment of 0 at infinite temperature? [3 points]

AT $T = \infty$ $\quad \beta = 0 \quad Z_{TOTAL} = 3$

$$\boxed{P = \frac{1}{3}}$$

(c) What is the average energy for this system? [3 points]

$$\bar{E} = \frac{1}{Z}\left(-\mu B e^{-\mu B \beta} + \mu B e^{\mu B \beta}\right)$$

$$\boxed{\langle \bar{E} \rangle = \frac{\mu B}{Z}\left(e^{\mu B \beta} - e^{-\mu B \beta}\right)}$$

(d) What is the Helmholtz free energy of this system? [3 points]

$$F = -kT \ln Z$$

$$\boxed{F = -kT \ln\left(1 + e^{-\mu B \beta} + e^{\mu B \beta}\right)}$$

(e) ~~What is the entropy of the~~ system? [3 points] ENTROPY?

WHAT IS THE APPROACH TO CALCULATE

$$\boxed{\text{USE} \quad \cancel{BS} \quad S = \frac{\partial F}{\partial T}}$$

7. Consider a system consisting of a single hydrogen atom/ion, which has two possible states: unoccupied with no electrons and occupied with one electron present, i.e. unoccupied state has zero energy with zero electrons and occupied state has energy -I and has one electron in it with a chemical potential of μ. Calculate the ratio of the probability of these two states. You may use $\mu=-kT\ln(V/NV_Q)$ to simplify the formula. Simplify such that there is only one exponential in the final ratio. You can assume that electrons behave like an ideal gas (PV=NkT). Use only P, V_Q, k, T, I to express the ratio. [15 points]

$E = 0 \quad n = 0 \leftarrow$ unoccu

$E = -I \quad n = 1 \leftarrow$ occupied

$$\frac{P_{un}}{P_{occ}} = \frac{1}{e^{-(-I-\mu)/kT}} = \frac{1}{e^{(I+\mu)/kT}}$$

$$= \frac{e^{-I/kT}}{e^{\mu/kT}}$$

$$\mu/kT = -\ln\left(\frac{V}{NV_Q}\right) = \ln\left(\frac{NV_Q}{V}\right) = \ln\left(\frac{PV_Q}{kT}\right)$$

$$NkT = PV$$

$$\frac{NkT}{P} = V$$

$$\text{RATIO:} = \frac{P_{\text{UNOCCUPIED}}}{P_{\text{OCCUPIED}}} = \boxed{\frac{kT}{PV_Q}\, e^{-I/kT}}$$

8. Calculate the densities of states for a 1D electron gas given that the allowed k vectors are k=(2π/L)*m with m being all integers. Hints: 1. calculate the density of states in therms of k, 2. use the energy formula (ħk)²/2m, 3. use N = 2* 2k*D(k) [10 points]

$$D(k) = \left(\frac{L}{2\pi}\right)$$

$$N(k) = 2 \cdot 2k \cdot \left(\frac{L}{2\pi}\right) = \frac{2L}{\pi} k$$

$$N = \frac{2L}{\pi}\left(\frac{2mE}{\hbar^2}\right)^{1/2} \qquad D\varepsilon = \frac{\partial N}{\partial \varepsilon}$$

$$\boxed{D(\varepsilon) = \frac{L}{\pi}\left(\frac{2m}{\hbar^2}\right)^{1/2} \varepsilon^{-1/2}}$$

3
4
8 coeff
9 − coeff