1. (a) Write the defining equation of a plane in Cartesian, cylindrical, and spherical coordinates
(b) Write the defining equation of a cylinder in Cartesian, cylindrical, and spherical coordinates
(c) Write the defining equation of a sphere in Cartesian, cylindrical, and spherical coordinates
(d) Write the defining equation of a cone in Cartesian, cylindrical, and spherical coordinates

2. Classify each of the following systems according as they are (i) holonomic or non-holonomic, (ii) scleronomic or rheonomic:
(a) a horizontal cylinder of radius $a$ rolling inside a perfectly horizontal cylinder of radius $b > a$;
(b) a cylinder rolling (and possibly sliding) down an inclined plane of angle $\phi$;
(c) a sphere rolling down another sphere which is rolling with uniform speed along a horizontal plane;
(d) a particle constrained to move along a line under the influence of a force which is inversely proportional to the square of its distance from a fixed point and a damping force proportional to the square of the speed;
(e) a particle sliding down the inner surface, with coefficient of friction $\mu$, of a paraboloid of revolution having its axis vertical and vertex downward.

3. From the book of Hand and Finch do Problem 2 of Chapter 1.

4. **Velocity and acceleration in polar coordinates**: Consider a particle moving on a plane. Introduce polar coordinates $\rho, \phi$ to describe the motion:

$$x = \rho \cos \phi, \quad y = \rho \sin \phi.$$  

The position of the particle is defined by

$$\vec{r} = x \hat{i} + y \hat{j}.$$  

(a) Find the unit vectors $\hat{u}_\rho, \hat{u}_\phi$ and express $\vec{r}$ in terms of them.
(b) Find the velocity of the particle in polar coordinates.
(c) Find the acceleration of the particle in polar coordinates.
5. **Velocity and acceleration in cylindrical coordinates**: Consider a particle moving in space. Introduce cylindrical coordinates \( \rho, \phi, z \) to describe the motion:

\[
x = \rho \cos \phi , \quad y = \rho \sin \phi , \quad z .
\]

The position of the particle is defined by

\[
\vec{r} = x \hat{i} + y \hat{j} + z \hat{k} .
\]

(a) Find the unit vectors \( \hat{u}_\rho, \hat{u}_\phi \) and express \( \vec{r} \) in terms of them and \( \hat{k} \).

(b) Find the velocity of the particle in cylindrical coordinates.

(c) Find the acceleration of the particle in cylindrical coordinates.

6. **Velocity and acceleration in spherical coordinates**: Consider a particle moving in space. Introduce spherical coordinates \( \rho, \phi, \theta \) to describe the motion:

\[
x = r \sin \theta \cos \phi , \quad y = r \sin \theta \sin \phi , \quad z = r \cos \theta .
\]

(a) Find the unit vectors \( \hat{u}_r, \hat{u}_\phi \) and \( \hat{u}_\theta \).

(b) Then show that the velocity and acceleration of the particle in spherical coordinates are given by

\[
\vec{v} = \dot{r} \hat{u}_r + r \sin \theta \dot{\phi} \hat{u}_\phi + r \dot{\theta} \hat{u}_\theta ,
\]

\[
\vec{a} = \left( \ddot{r} - r \ddot{\theta}^2 - r \dddot{\phi}^2 \sin^2 \theta \right) \hat{u}_r + \left( r \ddot{\phi} \sin \theta + 2 \dot{r} \dot{\theta} \sin \theta + 2 \dot{r} \ddot{\theta} \cos \theta \right) \hat{u}_\phi \\
+ \left( r \ddot{\theta} + 2 \dot{r} \dot{\theta} - r \dddot{\phi}^2 \sin \theta \cos \theta \right) \hat{u}_\theta .
\]

7. **Understanding notation**: Often, this instructor uses the symbol \( \partial / \partial \vec{r} \) as an alternative (and more intuitive) notation to \( \nabla \).

Let \( U(\vec{r}) = U(x, y, z) \) be a function and \( \vec{r} = \vec{r}(q) \). Verify that

\[
\frac{\partial U}{\partial \vec{r}} \cdot \frac{\partial \vec{r}}{\partial q} = \frac{\partial U}{\partial q} .
\]

8. **Curvilinear coordinates**: Consider a particle moving in space. Instead of the known coordinates (cartesian, spherical or cylindrical) we can describe the motion through
any arbitrarily chosen set of coordinates \( q_1, q_2, q_3 \). We can imagine a non-singular transformation that relates these coordinates to our familiar cartesian coordinates:

\[
x = x(q_1, q_2, q_3), \quad y = y(q_1, q_2, q_3), \quad z = z(q_1, q_2, q_3).
\]

(a) Show that the unit vectors \( \hat{u}_i, i = 1, 2, 3 \) are given by \( \hat{u}_i = \frac{\vec{e}_i}{|\vec{e}_i|} \) with

\[
\vec{e}_i = \frac{\partial \vec{r}}{\partial q_i}.
\]

(b) Show that a small displacement along the particle’s path can be expressed in the form

\[
d\vec{r} = h_1 dq_1 \hat{u}_1 + h_2 dq_2 \hat{u}_2 + h_3 dq_3 \hat{u}_3,
\]

and explain how the coefficients \( h_i, i = 1, 2, 3 \) are determined. Then express the velocity \( \vec{v} \) of the particle in terms of the coordinates \( q_1, q_2, q_3 \).

(c) Show that

\[
d\ell^2 = \sum_{i,j=1}^{3} g_{ij} dq_i dq_j,
\]

with

\[
g_{ij} = \vec{e}_i \cdot \vec{e}_j.
\]

Is there a relation between the two set of coefficients \( g_{ij} \) and \( h_i \)? Express the speed \( v \) of the particle in terms of the coordinates \( q_1, q_2, q_3 \).

(d) The coordinates \( q_1, q_2, q_3 \) are called orthogonal if \( g_{ij} = 0 \). Assume that this is the case and let \( \Phi \) be a function of \( q_1, q_2, q_3 \). Write down its differential. Then, comparing with

\[
d\Phi = \frac{\partial \Phi}{\partial \vec{r}} \cdot d\vec{r},
\]

derive an expression for the gradient \( \nabla \Phi \) in the \( q \)-coordinates.