Even and odd functions

- We define an even function such that $f(-x) = f(x)$
- We define an odd function such that $f(-x) = -f(x)$
- Example, $\sin x$ is an odd function because $\sin(-x) = -\sin x$
- Example, $\cos x$ is an even function because $\cos(-x) = \cos x$
- Now consider a Fourier series of a periodic, even function $f(x)$ ($f(-x) = f(x)$), over the interval $-\pi < x < \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

- Now consider the integrals to determine the coefficients, first $a_0$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi} \int_{0}^{\pi} f(x)dx$$
• Next the $a_n$ for finite $n$, we again use the fact that $f(x)$ is even, and also $\cos nx$ is even,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nxdx = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos nxdx$$

• Next we can show $b_n = 0$ when $f(x)$ is even,

$$b_n = \frac{1}{\pi} \left[ \int_{0}^{\pi} f(x) \sin nxdx - \int_{0}^{\pi} f(x) \sin nxdx \right] = 0$$
• We can also treat the odd case $f(-x) = -f(x)$, then

\[
a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx = 0
\]

\[
a_n = \frac{1}{\pi} \left[ \int_{0}^{\pi} f(x) \cos nx \, dx - \int_{0}^{\pi} f(x) \cos nx \, dx \right] = 0
\]

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx \, dx
\]
Example of an odd function

• The step function provides us an example of an odd function
• For the step function, we found $a_0 = 1$, so actually it is neither odd nor even, but if we define the step function as $f(x) = -1/2$ for $-\pi < x < 0$ and $f(x) = 1/2$ for $0 < x < \pi$, then $f(x)$ is odd and

\[ f(x) = \frac{2}{\pi} \left( \frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots \right) \]

• The point is that the $\cos nx$ terms vanish ($a_n = 0$ for all $n$)
• The advantage is that we could have predicted that $a_n = 0$ for all $n$ even before trying to do the integral
Another example: even function

- Find the expansion for \( f(x) = x^2 \) on the interval \(-\pi < x < \pi\), periodically repeating with period \(2\pi\).
- For this case, \( f(x) \) is clearly even since \( f(-x) = (-x)^2 = x^2 = f(x) \), hence \( b_n = 0 \) for each \( n \), and

\[
a_n = \frac{2}{\pi} \int_0^\pi x^2 \cos nx \, dx
\]

- We find \( a_0 = \frac{2\pi^2}{3} \).
- For the other \( n \), we evaluate the integral (homework!)

\[
\frac{2}{\pi} \int_0^\pi x^2 \cos nx \, dx = \frac{4}{n^2} (-1)^n
\]

\[
f(x) = \frac{\pi^2}{3} + 4 \left[ -\frac{\cos x}{1} + \frac{\cos 2x}{4} - \frac{\cos 3x}{9} + \ldots \right]
\]
Parseval’s theorem

• For a periodic function \( f(x) \) defined on \(-l < x < l\), we have

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}
\]

• The average of \([f(x)]^2\) is \(\frac{1}{2l} \int_{-l}^{l} [f(x)]^2 \, dx\)

• To obtain Parseval’s theorem, use the integrals we obtained before

\[
\frac{1}{2l} \int_{-l}^{l} \sin \frac{m\pi x}{l} \sin \frac{n\pi x}{l} \, dx = \frac{1}{2} \delta_{m,n}
\]

\[
\frac{1}{2l} \int_{-l}^{l} \cos \frac{m\pi x}{l} \cos \frac{n\pi x}{l} \, dx = \frac{1}{2} \delta_{m,n}
\]

\[
\frac{1}{2l} \int_{-l}^{l} \sin \frac{m\pi x}{l} \cos \frac{n\pi x}{l} \, dx = 0
\]
Parseval’s theorem continued

- Using the previous integrals, we find

\[
\frac{1}{2l} \int_{-l}^{l} [f(x)]^2 \, dx = \left( \frac{1}{2} a_0 \right)^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)
\]

- Example: Problem 5.8 and Problem 11.7
- Find the Fourier series for \( f(x) = 1 + x \) defined on \(-\pi < x < \pi\)

\[
a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (1 + x) \, dx = 2
\]

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (1 + x) \cos n x \, dx = 0
\]

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (1 + x) \sin n x \, dx = \frac{2(-1)^{n+1}}{n}
\]
Example of Parseval’s theorem continued

- Then Parseval’s theorem states,

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} (1 + x)^2 \, dx = 1 + \frac{1}{2} \sum_{n=1}^{\infty} \frac{4}{n^2} = 1 + 2 \sum_{n=1}^{\infty} \frac{1}{n^2}
\]

- Problem 11.8 asks us to evaluate \( \sum_{n=1}^{\infty} \frac{1}{n^2} \), and from Parseval’s theorem we see that,

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{1}{2} + \frac{1}{4\pi} \int_{-\pi}^{\pi} (1 + x)^2 \, dx = \frac{\pi^2}{6}
\]

- Might even use to compute \( \pi! \)

\[
\pi = \sqrt{6} \left[ \sum_{n=1}^{\infty} \frac{1}{n^2} \right]^{1/2}
\]
• Exact value of $\pi = 3.141592653589793$ (Correct to 16 digits... my computer using intrinsic functions got the digits after these incorrect)
• From serious on previous page, I got the following results:
  $10^4$ terms: 3.141497163947214
  $10^5$ terms: 3.141583104326456
  $10^6$ terms: 3.141591698660508
  $10^7$ terms: 3.141592558095902
• Correct to 7 digits for $10^7$ terms, and took $< 1$ second to compute
Parseval’s theorem for complex Fourier series

- When we average $|f(x)|^2 = f^*(x)f(x)$ over one period, we obtain $\sum_{n=-\infty}^{\infty} |c_n|^2$
- Proof in problem 3, for $f(x)$ periodic with periodicity $2\pi$ ($-\pi < x < \pi$)

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

- We use the orthogonality of the functions $e^{inx}$,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(n-m)x} \, dx = \delta_{m,n}$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f^*(x)f(x) \, dx = \frac{1}{2\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_m^* c_n \int_{-\pi}^{\pi} e^{i(m-n)x} \, dx = \sum_{n=-\infty}^{\infty} c_n^* c_n$$
• We can also average $[f(x)]^2$ using the complex series (contrast to averaging $|f(x)|^2 = f^*(x)f(x)$)

$$
\frac{1}{2\pi} \int_{-\pi}^{\pi} [f(x)]^2 \, dx = \frac{1}{2\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_m c_n \int_{-\pi}^{\pi} e^{i(m+n)x} \, dx = \sum_{n=-\infty}^{\infty} c_n c_{-n}
$$

• Consider the special case where $f(x)$ is real, then the expansion in complex Fourier series is

$$
f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} = c_0 + \sum_{n=1}^{\infty} \left( c_n e^{inx} + c_{-n} e^{-inx} \right)
$$

• Since $f(x)$ is real, the complex parts must cancel, so using the Euler formula
Problem 2 continued

\[
f(x) = c_0 + \sum_{n=1}^{\infty} (c_n + c_{-n}) \cos nx + \sum_{n=1}^{\infty} (ic_n - ic_{-n}) \sin nx
\]

- For the imaginary parts to go away, we require \( c_{-n} = c_n^* \)

\[
c_n + c_{-n} = c_n + c_n^* = 2 \text{Re}[c_n]
\]

\[
ic_n - ic_{-n} = ic_n - ic_n^* = -2 \text{Im}[c_n]
\]

- Then for real \( f(x) \), we obtain

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} [f(x)]^2 dx = \frac{1}{2\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_m c_n \int_{-\pi}^{\pi} e^{i(m+n)x} dx = \sum_{n=-\infty}^{\infty} c_n^* c_n
\]