Complex Fourier series

- Recalling the Euler formula, we can write

\[
\cos nx = \frac{e^{inx} + e^{-inx}}{2}
\]

\[
\sin nx = \frac{e^{inx} - e^{-inx}}{2i}
\]

- This suggests that instead of using \( \cos nx \) and \( \sin nx \), we might use \( e^{inx} \)

\[
f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}
\]

- To obtain the coefficients, multiply left and right sides by \( e^{-imb} \) and integrate over one period

\[
c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx
\]
Other intervals

- For a function with periodicity $2\pi$, we used functions with periodicity $2\pi$ for the expansion, namely $\cos x$, $\sin x$, or $e^{inx}$.
- Consider now a function with periodicity $2l$, so that $f(x + 2l) = f(x)$.
- We can see that $\sin \frac{n\pi x}{l}$ and $\cos \frac{n\pi x}{l}$ have the desired periodicity, as does $e^{\frac{inx}{l}}$.
- We then have the relations

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l}$$

- Or we might use the complex series,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{n\pi x}{l}}$$
The coefficients then are found from

\[ a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} \, dx \]

\[ b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} \, dx \]

\[ c_n = \frac{1}{2l} \int_{-l}^{l} f(x) e^{-inx} \, dx \]

We can see this is consistent with the case \( l = \pi \) we had before.
Dirichlet theorem

- While we can generate functions $f(x)$ with periodicity $l$, it is not clear yet whether the Fourier series converges to any function $f(x)$.
- Without proof, we state the Dirichlet theorem, which in essence shows that in most ordinary cases, the Fourier series will converge to the function $f(x)$.

Dirichlet theorem: If $f(x)$ is periodic, single valued, with a finite number of discontinuities and a finite number of maxima and minima, and if the average value of $f(x)$ is finite over the interval, then the Fourier series converges to $f(x)$ at all points where $f(x)$ is continuous, and to the midpoint anywhere there is a discontinuity.

- As an example, recall the function $f(x)$ with periodicity $2\pi$, defined from $-\pi$ to $\pi$ as $f(x) = 0$ for $-\pi < x < 0$ and $f(x) = 1$ for $0 < x < \pi$, we get the series (see Example 1 in Boas)

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \left( \frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots \right)$$
Convergence: practical example of step function

\[ f(x) = \frac{1}{2} + \frac{2}{\pi} \left( \sin \frac{x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots \right) \]

- We can plot this for different numbers of terms to see convergence for 1 term, 5 terms, and 25 terms