Singular matrices... does a matrix always have an inverse?

- No! Matrices need not have an inverse
- If a matrix does not have an inverse, it is not *invertible*, or it is called *singular*

Consider our expression for $M^{-1}$,

$$M^{-1} = \frac{1}{\det M} C^T$$

- $M$ is singular if $\det M = 0$

Geometric interpretation:
If we consider the rows (or columns) of $M$ to be vectors (consider a $3 \times 3$ matrix, the determinant is the volume $\Omega$ of a polyhedron described by the vectors. Then $\Omega = \det M = 0$ corresponds to the case where the three vectors lie all in a plane. This is related to the question of linear dependence/independence that we will address later.
• Linear transformations are important applications of linear algebra
• Coordinate transformations are important, including rotations
• Start with a vector \( \mathbf{r} = x \mathbf{i} + y \mathbf{j} \) and rotate by angle \( \phi \)
• After rotation, \( x \rightarrow x' \) and \( y \rightarrow y' \), or \( \mathbf{r} \rightarrow \mathbf{R} \)
• We can find the rotation matrix \( R(\phi) \) and the transformation

\[
\begin{pmatrix}
x' \\
y'
\end{pmatrix} = \begin{pmatrix}
\cos \phi & -\sin \phi \\
\sin \phi & \cos \phi
\end{pmatrix} \begin{pmatrix}
x \\
y
\end{pmatrix}
\]

• What is the inverse of this matrix? Think of the simplest idea
We could rotate back with $R^{-1}(\phi) = R(-\phi)$

$$R(\phi) = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}$$

$$R^{-1}(\phi) = R(-\phi) = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}$$

Try it!

$$R^{-1}(\phi)R(\phi) = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
Multiple rotations, and another (equivalent) picture

- It is easy to show that $R(\theta)R(\phi) = R(\theta + \phi)$ (see Problem 25, section 6).
- Moreover, $R(\gamma)R(\theta)R(\phi) = R(\gamma + \theta + \phi)$, etc.
- In another picture, we can imagine rotating the axes by an angle $\phi$, and leaving vector $\vec{r} = x\hat{i} + y\hat{j}$ fixed.
- In this picture, $x'$ and $y'$ define point in new coordinate system, and are given by

$$
\begin{pmatrix}
  x' \\
  y'
\end{pmatrix}
= 
\begin{pmatrix}
  \cos \phi & \sin \phi \\
  -\sin \phi & \cos \phi
\end{pmatrix}
\begin{pmatrix}
  x \\
  y
\end{pmatrix}
$$

- Both of these pictures are often used... In one rotating the vector, and the other rotating the coordinate system.
- Rotation is example of a linear operator describing a linear transformation.
- It is also an example of an orthogonal transformation.
A linear combination is simply addition of two vectors, $\vec{A} + \vec{B}$
A *linear combination* is simply addition of two vectors, $\vec{A} + \vec{B}$.

A *scalar function* of a vector is $f(\vec{r})$. 
A linear combination is simply addition of two vectors, $\vec{A} + \vec{B}$

A scalar function of a vector is $f(\vec{r})$

A vector function of a vector is $\vec{F}(\vec{r})$
A *linear combination* is simply addition of two vectors, $\vec{A} + \vec{B}$

A *scalar function* of a vector is $f(\vec{r})$

A *vector function* of a vector is $\vec{F}(\vec{r})$

An operator acts on a scalar, vector, etc. and results in a new scalar, vector, etc.
Examples: Linear functions and operators

- To be a *linear function*, we require that $f(\vec{r}_1 + \vec{r}_2) = f(\vec{r}_1) + f(\vec{r}_2)$ and that $f(a\vec{r}) = af(\vec{r})$ where $a$ is a scalar.

- Example: A scalar function of a vector, problem 3, section 7
  Is the following scalar function of the vector $\vec{r}$ linear?

  $$ f(\vec{r}) = \vec{r} \cdot \vec{r} $$

  We see that for $\vec{r} = \vec{r}_1 + \vec{r}_2$,

  $$ f(\vec{r}) = f(\vec{r}_1 + \vec{r}_2) = (\vec{r}_1 + \vec{r}_2) \cdot (\vec{r}_1 + \vec{r}_2) = \vec{r}_1 \cdot \vec{r}_1 + \vec{r}_2 \cdot \vec{r}_2 + 2\vec{r}_1 \cdot \vec{r}_2 $$

  But, for $f(\vec{r}_1) + f(\vec{r}_2)$

  $$ f(\vec{r}_1) + f(\vec{r}_2) = \vec{r}_1 \cdot \vec{r}_1 + \vec{r}_2 \cdot \vec{r}_2 $$

  Thus it is not linear. The other requirement also is not met.
Examples: Linear functions and operators

• To be a *linear function*, we require that
  \( \vec{F}(\vec{r}_1 + \vec{r}_2) = \vec{F}(\vec{r}_1) + \vec{F}(\vec{r}_2) \) and that
  \( \vec{F}(a\vec{r}) = a\vec{F}(\vec{r}) \) where \( a \) is a scalar.

• Example: A vector function of a vector, problem 5, section 7
  Is the following function of the vector \( \vec{r} \) linear? The \( \vec{A} \) is a given vector.

  \[
  \vec{F}(\vec{r}) = \vec{A} \times \vec{r}
  \]

  It is obvious that
  \[
  \vec{A} \times (\vec{r}_1 + \vec{r}_2) = \vec{A} \times \vec{r}_1 + \vec{A} \times \vec{r}_2,
  \]
  and also
  \[
  \vec{A} \times (a\vec{r}) = a\vec{A} \times \vec{r}
  \]
  Therefore, \( \vec{F}(\vec{r}) = \vec{A} \times \vec{r} \) is a linear vector function

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A linear operator and the properties $O(A + B) = O(A) + O(B)$ and $O(kA) = kO(A)$, where $k$ is just a number.

Example: Is $\frac{d}{dx}$ a linear operator?

$$\frac{d}{dx} [f(x) + g(x)] = \frac{df}{dx} + \frac{dg}{dx}$$

$$\frac{d}{dx} kf(x) = k \frac{df}{dx}$$

So yes, $\frac{d}{dx}$ is a linear operator.

The elements $A$ and $B$ above can be numbers, functions, vectors, etc.

In quantum mechanics we consider operators in some vector space, continuous or discrete.
Orthogonal transformation

- Consider a transformation that does not change the length of the vector. This is an orthogonal transformation

\[(x')^2 + (y')^2 = x^2 + y^2\]

- Rotation is a good example of an orthogonal transformation

\[(x')^2 + (y')^2 = (x' \ y') \begin{pmatrix} x' \\ y' \end{pmatrix}\]

We just use,

\[
\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}
\]

\[
\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\]

- Note that \(R(\phi)\) is an example of a linear operator
Then we get for $(x')^2 + (y')^2 = (x'\ y') \left( \begin{array}{c} x' \\ y' \end{array} \right)$,

\[
\begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^2
\]

This is obvious because we just have $R^{-1}(\phi)R(\phi)$,

\[
\begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]
• We also observe that $R^{-1}(\phi) = R^T(\phi)$.

$$R^{-1}(\phi) = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}^T = R^T(\phi)$$

• For any orthogonal transformation $M$, we find $M^T = M^{-1}$
• We also find for orthogonal transformations $\det M = \pm 1$

We can show this since $M^T M = I$, and then

$$\det (M^T M) = (\det M^T)(\det M) = (\det M)^2 = 1$$

• If $\det M = 1$, purely a rotation
• If $\det M = -1$, either a reflection/inversion
Inversions/reflections

• Another example of orthogonal transformation
• Simplest example, reflection across $x = 0$ ($x \rightarrow -x$ and $y \rightarrow y$)

\[ M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \]

• Here we see $\det M = -1$ as expected for pure reflection
• $M^{-1} = M^T = M$ (Try it!)

\[ M^T M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \]
More general inversions

- We might want to invert across some line other than \( x = 0 \) or \( y = 0 \).
- There is a straightforward way to do this combining rotation of coordinate axes, inversion, and then another rotation back to original system.
- Determine the transformation matrix \( M \) for inversion across the line \( y = -x \).

First rotate axes by an angle \( \phi = -\frac{\pi}{4} \). Next inversion across new axis \( y = 0 \). Finally rotate coordinate axes back by \( \phi = \frac{\pi}{4} \).

\[
M = \begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix} \begin{pmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix} = \begin{pmatrix}
0 & -1 \\
-1 & 0
\end{pmatrix}
\]

This matrix has \( M^T = M^{-1} = M \) and \( \text{det} M = -1 \) for reflections.

Further check: \( M \) should leave a vector \( -\hat{i} + \hat{j} \) unchanged.

\[
\begin{pmatrix}
0 & -1 \\
-1 & 0
\end{pmatrix} \begin{pmatrix}
-1 \\
1
\end{pmatrix} = \begin{pmatrix}
-1 \\
1
\end{pmatrix}
\]
Determining what transformation a matrix represents

- We should be able to figure out what kind of transformation a matrix $M$ represents
- If $\det M = 1$, $M$ is a rotation, but if $\det M = -1$, $M$ is a reflection (If $\det M$ is not $\pm 1$, it is not an orthogonal transformation)
- Example: Problem 23, section 7:

Determine if the matrix $M$ is orthogonal, and whether it is a rotation or reflection, and lastly find the rotation angle or line of reflection

$$M = \begin{pmatrix} -\sqrt{3}/2 & 1/2 \\ -1/2 & -\sqrt{3}/2 \end{pmatrix}$$

First, we see that $\det M = \begin{vmatrix} -\sqrt{3}/2 & 1/2 \\ -1/2 & -\sqrt{3}/2 \end{vmatrix} = 1$. This corresponds to a rotation. We have $\cos \phi = -\sqrt{3}/2$ and $\sin \phi = -1/2$. Hence, $\phi = \frac{7\pi}{6}$. 
Another example...

- Example: Problem 26, section 7:
  Determine if the matrix $M$ is orthogonal, and whether it is a rotation or reflection, and lastly find the rotation angle or line of reflection

  \[
  M = \begin{pmatrix}
  \frac{3}{5} & \frac{4}{5} \\
  \frac{4}{5} & -\frac{3}{5}
  \end{pmatrix}
  \]

  First we see that $\det M = \begin{vmatrix}
  \frac{3}{5} & \frac{4}{5} \\
  \frac{4}{5} & -\frac{3}{5}
  \end{vmatrix} = -1$, so this is a reflection.

  To find the line of the reflection, find a vector that is unchanged by the transformation.

  \[
  \begin{pmatrix}
  \frac{3}{5} & \frac{4}{5} \\
  \frac{4}{5} & -\frac{3}{5}
  \end{pmatrix}
  \begin{pmatrix}
  x \\
  y
  \end{pmatrix}
  =
  \begin{pmatrix}
  x \\
  y
  \end{pmatrix}
  \]

  We can also write this as,

  \[
  \begin{pmatrix}
  -\frac{2}{5} & \frac{4}{5} \\
  \frac{4}{5} & -\frac{8}{5}
  \end{pmatrix}
  \begin{pmatrix}
  x \\
  y
  \end{pmatrix}
  = 0
  \]
Both of the equations this represents are identical, so we find easily that \( y = \frac{1}{2}x \), so a line \( 2\hat{i} + \hat{j} \) describes the line of reflection.