Steady-state temperature in a sphere

- Consider a sphere of radius \( r = 1 \), with the temperature \( T = 100 \) on the top half \((z > 0 \text{ or } 0 < \theta < \pi/2)\) and \( T = 0 \) on the bottom half \((z < 0 \text{ or } \pi/2 < \theta < \pi)\).

- We know that our solution is a solution to Laplace equation \( \nabla^2 T = 0 \) most conveniently in spherical coordinates

\[
T(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} r^l P_l^m(\cos \theta) \left[ a_{lm} \cos m\phi + b_{lm} \sin m\phi \right]
\]

- Based on the problem, we see there is no \( \phi \) dependence, so we only require \( m = 0 \) and \( \cos m\phi = 1 \), so we simplify

\[
T(r, \theta) = \sum_{l=0}^{\infty} c_l r^l P_l^m=0(\cos \theta)
\]
Steady-state temperature in a sphere

- Solution is a series in the Legendre polynomials! At \( r = 1 \) we know \( T(r = 1, \theta) \)

\[
T(r = 1, \theta) = \sum_{l=0}^{\infty} c_l P_l(\cos \theta)
\]

- Find the coefficients \( c_l \)!

- With \( T(r = 1, x = \cos \theta) = 100f(x) \), we have \( f(x) = 0 \) for \(-1 < x < 0\) and \( f(x) = 1 \) for \( 0 < x < 1 \), and then

\[
c_l = 100 \left( \frac{2l + 1}{2} \right) \int_{-1}^{1} f(x)P_l(x)dx
\]

\[
c_0 = 100 \left( \frac{1}{2} \right) \int_{0}^{1} dx = 100 \left( \frac{1}{2} \right)
\]

\[
c_1 = 100 \left( \frac{3}{2} \right) \int_{0}^{1} xdx = 100 \left( \frac{3}{4} \right)
\]
Steady-state temperature in a sphere, continued

\[ c_2 = 100 \left( \frac{5}{2} \right) \int_0^1 \left( \frac{3}{2} x^2 - \frac{1}{2} \right) \, dx = 0 \]

\[ c_3 = 100 \left( \frac{7}{2} \right) \int_0^1 \left( \frac{5}{2} x^3 - \frac{3}{2} x \right) \, dx = 100 \left( -\frac{7}{16} \right) \]

- The integral for \( c_4 = 0 \) ... we can even see this by symmetry... so we can go to \( c_5 \)

\[ c_5 = 100 \left( \frac{11}{2} \right) \int_0^1 \left( \frac{1}{8} \right) \left( 63x^5 - 70x^3 + 15x \right) \, dx = 100 \left( \frac{11}{32} \right) \]
Now that we have our Legendre coefficients, $c_0 = 100 \left( \frac{1}{2} \right)$, $c_1 = 100 \left( \frac{3}{4} \right)$, $c_2 = 0$, $c_3 = 100 \left( -\frac{7}{16} \right)$, $c_4 = 0$, $c_5 = 100 \left( \frac{11}{32} \right)$, etc., we can write a series solution

$$T(r, \theta) = \sum_{l=0}^{\infty} c_l r^l P_l(\cos \theta)$$

$$T(r, \theta) = 100 \left[ \frac{1}{2} + \frac{3}{4} r \cos \theta - \frac{7}{16} r^3 \left( \frac{5}{2} \cos^3 \theta - \frac{3}{2} \cos \theta \right) + \ldots \right]$$

Notice that we write $T(r, \theta)$ since we determined from the beginning that the solution is independent of $\phi$.

It is crucial to remember that we solved this only for $r < 1$! Our solution does not apply outside of this region.