The orthogonality over the interval $-1 < x < 1$ can be used to make a series expansion of a function $f(x)$ over the same interval

$$f(x) = \sum_{l=0}^{\infty} c_l P_l(x)$$

We use the orthogonality of the Legendre functions to find integrals that determine the $c_l$

$$c_l = \frac{2l + 1}{2} \int_{-1}^{1} f(x) P_l(x) \, dx$$
Recurrence relations

• The recurrence relations below sometimes come in handy

\[ lP_l(x) = (2l - 1)xP_{l-1}(x) - (l - 1)P_{l-2}(x) \]

\[ xP'_l(x) - P'_{l-1}(x) = lP_l(x) \]

\[ P'_l(x) - xP'_{l-1}(x) = lP_{l-1}(x) \]

\[ (1 - x^2)P'_l(x) = lP_{l-1}(x) - lxP_l(x) \]

\[ (2l + 1)P_l(x) = P'_{l+1}(x) - P'_{l-1}(x) \]

• Notice first recursion relation implies, with \( P_0(x) = 1 \) and \( P_1(x) = x \), that highest power of \( P_l(x) \) is \( x^l \)
In electrostatics and gravitation, we see scalar potentials of the form \( V = \frac{K}{d} \).

- Take 
  \[ d = |\vec{R} - \vec{r}| = \sqrt{R^2 - 2Rr \cos \theta + r^2} = R\sqrt{1 - 2\frac{r}{R} \cos \theta + \left(\frac{r}{R}\right)^2} \]

- Use \( h = \frac{r}{R} \) and \( x = \cos \theta \), and then we see we have the generating function:

\[
V = \frac{K}{R} \left(1 - 2hx + h^2\right)^{-1/2} = \frac{K}{R} \sum_{l=1}^{\infty} h^l P_l(x)
\]

- Then in terms of the \( r \) and \( \theta \) variables, we have

\[
V = K \sum_{l=0}^{\infty} \frac{r^l P_l(\cos \theta)}{R^{l+1}}
\]
Multipole expansion

• If we have make charges $q_i$ at different coordinates $\vec{r}_i$, then we can use this to find the electrostatic potential at $\vec{R}$

$$V = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \sum_i q_i r_i^l P_l(\cos \theta_i) \frac{1}{R^{l+1}}$$

• Or if we have a continuous distribution $\rho(\vec{r})$,

$$V = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \int \int \int r^l P_l(\cos \theta) \rho d\tau \frac{1}{R^{l+1}}$$

• Lowest order term $l = 0$, is just the total charge, $V \propto \frac{1}{R}$

$$Q = \int \int \int \rho d\tau$$
Multipole expansion, continued

• Next order term $l = 1$ is the dipole moment, $V \propto \frac{1}{R^2}$

$$p = \int \int \int r \cos \theta \rho \, d\tau$$

• Writing both the $l = 0$ (monopole) and $l = 1$ (dipole) terms, we have

$$V = \frac{1}{4\pi\epsilon_0} \left[ \frac{Q}{R} + \frac{p}{R^2} + \ldots \right]$$

• Higher order terms take into account more details of the distribution with contributions that fall off faster with increasing $R$

• For example, the quadrupole moments contribute a potential $\propto \frac{1}{R^3}$