Homework 5
PHZ 3113
Due Monday, February 15, 2010
Chapter 4-5

1. Recall in class we showed how to find the average kinetic energy of a gas particle when the gas is at constant temperature $T$. In particular, we found $\langle \frac{1}{2}mv^2 \rangle = \frac{3}{2}k_BT$. The integrals in this problem are quite similar.

Consider a pendulum of length $L$ and mass $m$ moving in one-dimension in the gravitational field of the Earth with $g = 9.81 m/s^2$. The displacement of the pendulum from equilibrium can be described by an angle $\theta$. The potential energy of the pendulum can be shown to be $V(\theta) = -mgL(\cos \theta - 1)$.

a) Show that for small angles $\theta$, we can approximate the potential energy by $V(\theta) = \frac{mgL}{2}\theta^2$ and hence the system is a simple harmonic oscillator.

b) Now imagine that the pendulum is bombarded with gas atoms at a temperature $T$. If the mass $m$ of the pendulum is small, and we have a very sensitive apparatus, we might just be able to see that the gas atoms cause the pendulum to be slightly displaced from its equilibrium position $\theta = 0$. We would find that the average displacement $\langle \theta \rangle = 0$. However, we would find that $\langle \theta^2 \rangle$ is measurable.

The probability $P(\theta)d\theta$ of observing an angle between $\theta$ and $\theta + d\theta$ is given by,

$$P(\theta)d\theta = \frac{\sqrt{mgL}}{2\pi k_BT}e^{-\frac{mgL}{2k_BT}\theta^2}d\theta$$

First demonstrate that $\int_{-\infty}^{\infty} P(\theta)d\theta = 1$. Note that it is permissible to take the limits this way because the probability for very large displacements is quite small. Next, show that the average potential energy is found to be $\frac{mgL}{2}\langle \theta^2 \rangle = \frac{1}{2}k_BT$. (Hint: To show this, compute $\int_{-\infty}^{\infty} P(\theta)\theta^2 d\theta$ using the techniques/results found in lecture)

2. We showed briefly in class how to use Legendre transformations to write various quantities in thermodynamics. Take as our starting point the energy function $U(S,V,N)$, where $S$ is the entropy, $V$ is the volume, and $N$ is the particle number. We then have the differential,

$$dU = \frac{\partial U}{\partial S}dS + \frac{\partial U}{\partial V}dV + \frac{\partial U}{\partial N}dN$$
If we then define $T = \frac{\partial U}{\partial S}$, $p = -\frac{\partial U}{\partial V}$, and $\mu = \frac{\partial U}{\partial N}$ we have

$$dU = TdS - pdV + \mu dN$$

Consider the Legendre transformations $F = U - TS$, $G = F + pV$, $W = G + TS$, and $\Omega = F - \mu N$. Find expressions for the total differentials $dF$, $dG$, $dW$, and $d\Omega$. You should be able to determine what the independent parameters are for each of these functions. For example, above we see $U(S,V,N)$, and we showed in class $F(T,V,N)$. Finally, from the total differentials you obtained, show that we can also define the temperature $T = \frac{\partial W}{\partial S}$, and the pressure as $p = -\frac{\partial \Omega}{\partial V}$.

3. Boas, Chapter 4, Section 12, Problem 3
4. Boas, Chapter 5, Section 2, Problem 40
5. Boas, Chapter 5, Section 3, Problem 6
6. Boas, Chapter 5, Section 4, Problem 14