1. a) Show that the basis functions $e_n(x) = \frac{1}{\sqrt{2}} e^{inx}$ with $n = 0, \pm 1, \pm 2, \ldots$ forms an orthonormal set using the definition for the inner product as,

$$\langle m|n \rangle = \int_{-1}^{1} e^{*}_m(x) e_n(x) \, dx$$

In other words, show that $\langle m|n \rangle = \delta_{m,n}$.

b) This orthonormal basis is complete. This means we can use the basis to represent any single-valued function with a finite number of discontinuities. For the function $f(x) = \exp(x)$ defined from $-1 \leq x \leq 1$, find the coefficients $f_n$ in the expansion

$$f(x) = \sum_{n=-\infty}^{\infty} f_n e_n(x)$$

c) A basis can also be used to represent an operator $\hat{\Omega}$ as a matrix. The representation in the orthonormal basis above is given by

$$\Omega_{mn} = \langle m|\hat{\Omega}|n \rangle$$

Determine the matrix representation of the operator $\hat{\Omega} = \frac{d^2}{dx^2}$. What operator in quantum mechanics is this related to?

2. Consider the Helmholtz equation

$$\frac{d^2 f(x)}{dx^2} + k^2 f(x) = 0$$

a) Find all solutions $f_n(x)$ to the Helmholtz equation that satisfy the boundary conditions $\frac{df(x)}{dx}|_{x=0} = 0$ and $f(x = L) = 0$

b) Show that these functions once normalized form an orthonormal basis satisfying

$$\langle m|n \rangle = \int_{0}^{L} f^*_m(x) f_n(x) \, dx = \delta_{m,n}$$
3. Consider the diffusion or heat flow equation in two spatial dimensions

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u = \frac{1}{\alpha^2} \frac{\partial u}{\partial t}
\]

a) Use the method of separation of variables and take \( u(x, y, t) = F(x, y)T(t) \) to show that this equation can be reduced to two ordinary differential equations

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) F(x, y) + k^2 F(x, y) = 0
\]

and

\[
\frac{dT}{dt} = -k^2\alpha^2 T
\]

b) Find the general set of solutions subject to the boundary conditions \( u(x = 0, y, t) = 0, u(x = L, y, t) = 0, u(x, y = 0, t) = 0, u(x, y = L, t) = 0 \) (Hint: You might want to start by first applying separation of variables to the spatial equation, taking \( F(x, y) = X(x)Y(y) \)).

4. This problem is an example of a boundary-value problem that also incorporates Green’s functions. First, recall that we can write a function \( u(\mathbf{r}, t) \) as a Fourier transform

\[
u(\mathbf{r}, t) = \int_{-\infty}^{\infty} u(\mathbf{k}, t) \exp(i\mathbf{k} \cdot \mathbf{r}) \frac{d^3 \mathbf{k}}{(2\pi)^3}
\]

a) Directly substitute the Fourier transform above into the diffusion equation,

\[\nabla^2 u = \frac{1}{\alpha^2} \frac{\partial u}{\partial t}\]

and show that the solution is given by,

\[u(\mathbf{k}, t) = u_0 \exp(-\alpha^2 k^2 t)\]

b) Recall that the inverse transform is given by

\[u(\mathbf{k}, t) = \int_{-\infty}^{\infty} u(\mathbf{r}, t) \exp(-i\mathbf{k} \cdot \mathbf{r}) d^3 \mathbf{r}\]

If \( u(\mathbf{r}, t = 0) = \delta(x - x')\delta(y - y')\delta(z - z') \) (these are Dirac \( \delta \)-functions) is the \( t = 0 \) density, verify that \( u(\mathbf{k}, t = 0) = \exp(-i\mathbf{k} \cdot \mathbf{r}') \), with \( \mathbf{r}' = x'\hat{i} + y'\hat{j} + z'\hat{k} \).
c) Using parts a) and b) above, show that given these initial conditions, that the time-dependent density is

\[ u(x, y, z, t) = \frac{1}{(4\pi\alpha^2 t)^{3/2}} \exp \left[ -\frac{(x-x')^2 + (y-y')^2 + (z-z')^2}{4\alpha^2 t} \right] \]

\[ d) \text{ Consider two-dimensional diffusion in the } x - y \text{ plane (i.e. ignore } z \text{ in the above results). At } t = 0, \text{ the density is given by } u(x, y, t = 0) = \delta(x-a)\delta(y-a), \text{ where } a \text{ is positive so the initial density is in the first quadrant (see sketch). The plane at } x = 0 \text{ is completely reflecting, so it has the boundary condition } \frac{\partial u}{\partial x} \bigg|_{x=0} = 0. \text{ The plane at } y = 0 \text{ is completely absorbing, so it has the boundary condition } u(x, y = 0, t) = 0. \text{ Use the Green’s functions above to solve this problem. (Hint: Use the method of images to satisfy the boundary conditions).} \]