Chapter 9 – Center of mass and linear momentum

I. The center of mass

- System of particles / - Solid body

II. Newton's Second law for a system of particles

III. Linear Momentum

- System of particles / - Conservation

IV. Collision and impulse

- Single collision / - Series of collisions

V. Momentum and kinetic energy in collisions

VI. Inelastic collisions in 1D

-Completely inelastic collision/ Velocity of COM

VII. Elastic collisions in 1D

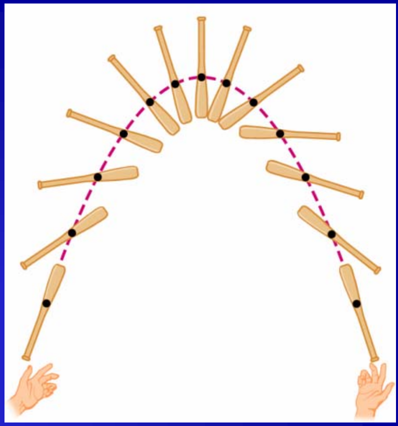
VIII. Collisions in 2D

IX. Systems with varying mass

X. External forces and internal energy changes

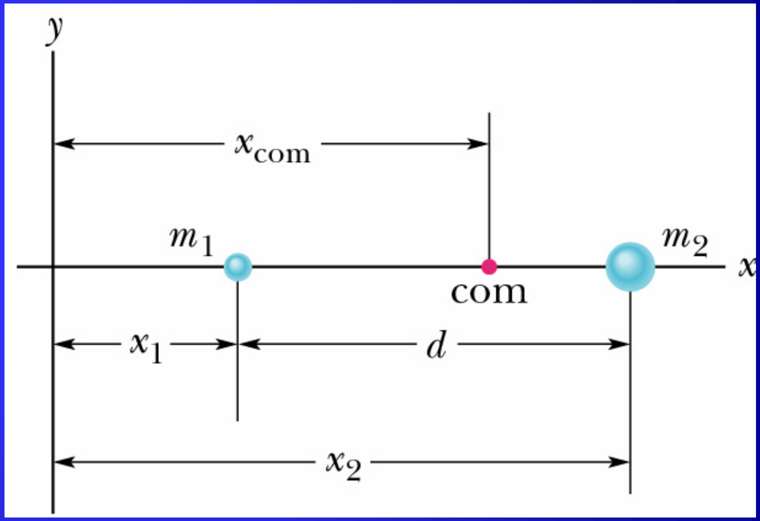
I. Center of mass

The center of mass of a body or a system of bodies is a point that moves as though all the mass were concentrated there and all external forces were applied there.



- System of particles:

General: $$x_{com} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{m_1 x_1 + m_2 x_2}{M}$$



M = total mass of the system

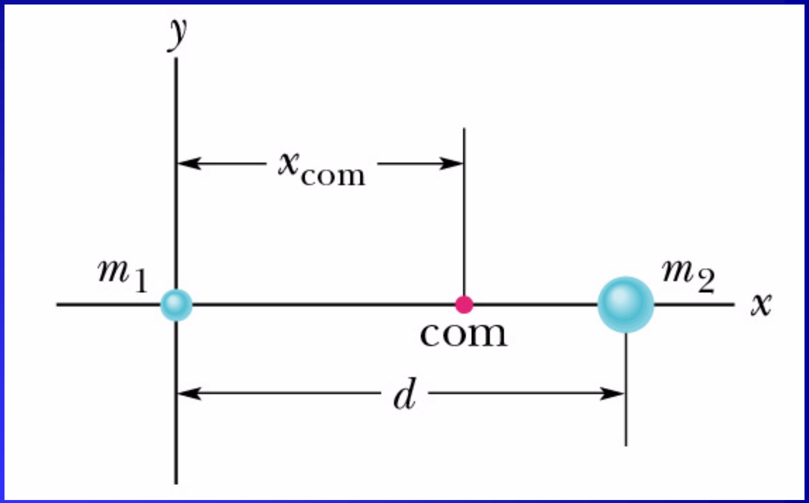
- The center of mass lies somewhere between the two particles.

- Choice of the reference origin is arbitrary → Shift of the coordinate system but center of mass is still at the same relative distance from each particle.

I. Center of mass

- System of particles:

$$x_{com} = \frac{m_2}{m_1 + m_2} d$$



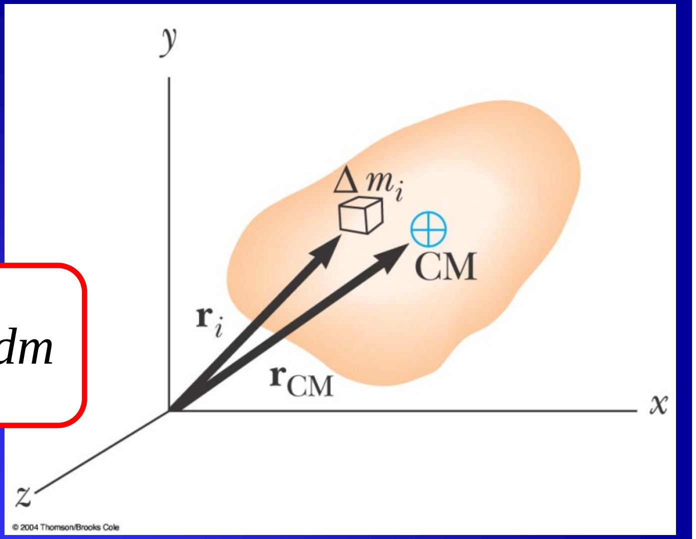
Origin of reference system coincides with m_1

3D:

$$x_{com} = \frac{1}{M} \sum_{i=1}^{n} m_i x_i \qquad y_{com} = \frac{1}{M} \sum_{i=1}^{n} m_i y_i \qquad z_{com} = \frac{1}{M} \sum_{i=1}^{n} m_i z_i$$

$$\vec{r}_{com} = \frac{1}{M} \sum_{i=1}^{n} m_i \vec{r}_i$$

- Solid bodies: Continuous distribution of matter. Particles = dm (differential mass elements).



3D:

$$x_{com} = \frac{1}{M}\int x\ dm \qquad y_{com} = \frac{1}{M}\int y\ dm \qquad z_{com} = \frac{1}{M}\int z\ dm$$

M = mass of the object

Assumption: Uniform objects → uniform density

$$\rho = \frac{M}{V} \rightarrow dm = \rho\ dV$$

Volume density

$$x_{com} = \frac{1}{V}\int x\ dV \qquad y_{com} = \frac{1}{V}\int y\ dV \qquad z_{com} = \frac{1}{V}\int z\ dV$$

Linear density: λ = M / L → dm = λ dx Surface density: σ = M / A → dm = σ dA

The center of mass of an object with a point, line or plane of symmetry lies on that point, line or plane.

The center of mass of an object does not need to lie within the object.

Examples: doughnut, horseshoe

Problem solving tactics:

(1) Use object's symmetry.
(2) If possible, divide object in several parts. Treat each of these parts as a particle located at its own center of mass.
(3) Chose your axes wisely. Use one particle of the system as origin of your reference system or let the symmetry lines be your axis.

II. Newton's second law for a system of particles

Motion of the center of mass: It moves as a particle whose mass is equal to the total mass of the system.

$$\vec{F}_{net} = M\vec{a}_{com}$$

- F_{net} is the **net of all external forces** that act on the system. Internal forces (from one part of the system to another are not included).

- <u>Closed system</u>: no mass enters or leaves the system during movement. (M=total mass of system).

- a_{com} is the acceleration of the system's center of mass.

$$F_{net,x} = Ma_{com,x} \qquad F_{net,y} = Ma_{com,y} \qquad F_{net,z} = Ma_{com,z}$$

Proof:

$$M\vec{r}_{com} = m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \ldots + m_n\vec{r}_n$$

$$M\frac{d\vec{r}_{com}}{dt} = M\vec{v}_{com} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \ldots + m_n\vec{v}_n$$

$$M\frac{d^2\vec{r}_{com}}{dt^2} = M\frac{d\vec{v}}{dt} = M\vec{a}_{com} = m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \ldots + m_n\vec{a}_n = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots + \vec{F}_n \qquad (*)$$

(*) includes forces that the particles of the system exert on each other (internal forces) and forces exerted on the particles from outside the system (external).

Newton's third law → internal forces from third-law force pairs cancel out in the sum (*) → Only external forces.

III. Linear momentum

- Vector magnitude.

Linear momentum of a particle: $\vec{p} = m\vec{v}$

The time rate of change of the momentum of a particle is equal to the net force acting on the particle and it is in the direction of that force.

$$\vec{F}_{net} = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt} = m\vec{a}$$

Equivalent to Newton's second law.

- System of particles:

The total linear moment $\vec{P}$ is the vector sum of the individual particle's linear momenta.

$$\vec{P} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + ... + \vec{p}_n = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + ... + m_n\vec{v}_n$$

$$\vec{P} = M\vec{v}_{com}$$

The linear momentum of a system of particles is equal to the product of the total mass M of the system and the velocity of the center of mass.

$$\frac{d\vec{P}}{dt} = M\frac{d\vec{v}_{com}}{dt} = M\vec{a}_{com} \rightarrow \vec{F}_{net} = \frac{d\vec{P}}{dt}$$

Net external force acting on the system.

- Conservation:

If no external force acts on a closed, isolated system of particles, the total linear momentum $\vec{P}$ of the system cannot change.

$$\vec{P} = cte \qquad (Closed, isolated\ system)$$

$$\vec{F}_{net} = 0 = \frac{d\vec{P}}{dt} \rightarrow \vec{P}_f = \vec{P}_i$$

Closed: no matter passes through the system boundary in any direction.

Isolated: the net external force acting on the system is zero. If it is not isolated, each component of the linear momentum is conserved separately if the corresponding component of the net external force is zero.

If the component of the net external force on a closed system is zero along an axis → component of the linear momentum along that axis cannot change.

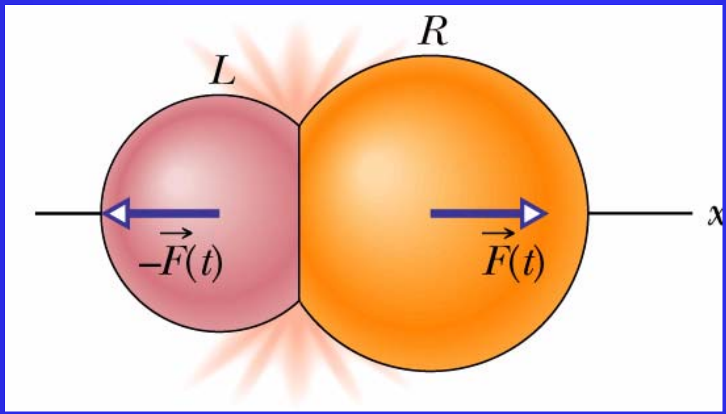
The momentum is constant if no external forces act on a closed particle system. Internal forces can change the linear momentum of portions of the system, but they cannot change the total linear momentum of the entire system.

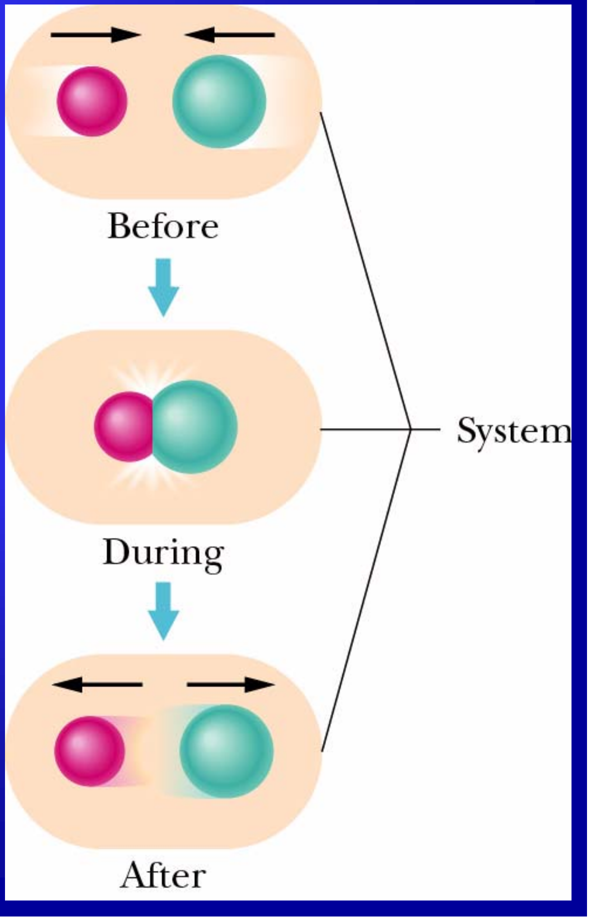
IV. Collision and impulse

Collision: isolated event in which two or more bodies exert relatively strong forces on each other for a relatively short time.

Impulse:

- Measures the strength and duration of the collision force
- Vector magnitude.





Third law force pair

$F_R = - F_L \rightarrow J_R = - J_L$

- Single collision

$$\vec{F} = \frac{d\vec{p}}{dt} \rightarrow d\vec{p} = \vec{F}(t)dt \rightarrow \int_{\vec{p}_i}^{\vec{p}_f} d\vec{p} = \int_{t_i}^{t_f} \vec{F}(t)dt$$

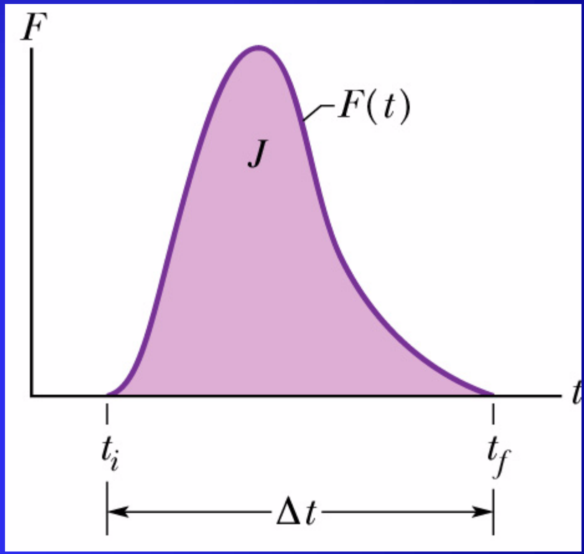
$$\boxed{\vec{J} = \int_{t_i}^{t_f} \vec{F}(t)dt = \vec{p}_f - \vec{p}_i = \Delta\vec{p}}$$

- Impulse-linear momentum theorem

The change in the linear momentum of a body in a collision is equal to the impulse that acts on that body.

$$\Delta\vec{p} = \vec{p}_f - \vec{p}_i = \vec{J}$$

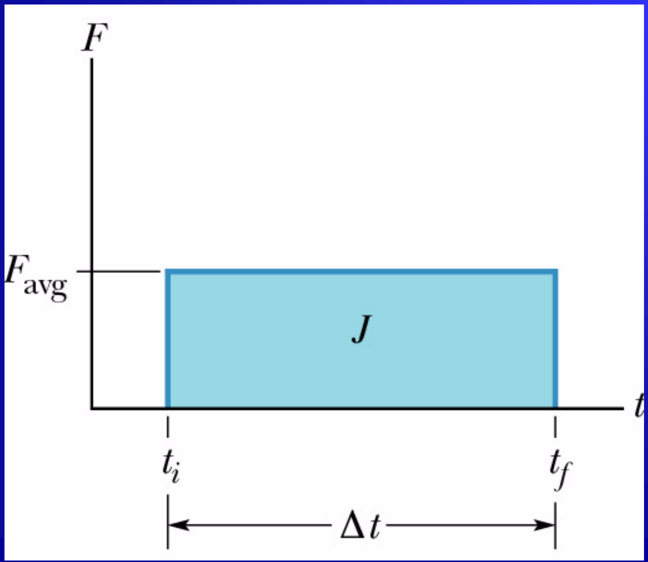
Units: kg m/s



$$p_{fx} - p_{ix} = \Delta p_x = J_x$$

$$p_{fy} - p_{iy} = \Delta p_y = J_y$$

$$p_{fz} - p_{iz} = \Delta p_z = J_z$$



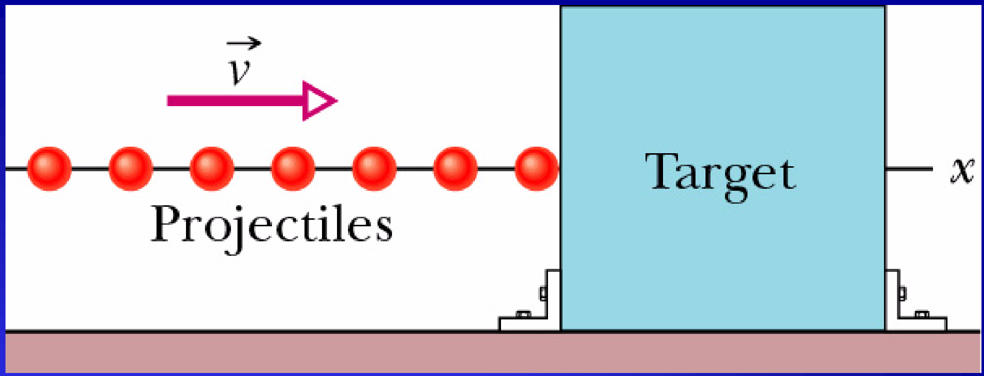
F_{avg} such that:

Area under F(t)-Δt curve = Area under F_{avg}- Δt

$$J = F_{avg}\Delta t$$

- Series of collisions

Target fixed in place n-projectiles → n Δp = Total change in linear momentum (projectiles)



Impulse on the target:

$$J_{target} = -J_{projectiles} = -n \cdot \Delta p$$

J and Δp have opposite directions, $p_f < p_i$ → Δp left → J to the right.

$$F_{avg} = \frac{J}{\Delta t} = \frac{-n}{\Delta t}\Delta p = \frac{-n}{\Delta t} m \Delta v$$

$n/\Delta t$ → Rate at which the projectiles collide with the target.

$$\Delta m = nm \ in \ \Delta t \rightarrow F_{avg} = -\frac{\Delta m}{\Delta t}\Delta v$$

$\Delta m/\Delta t$ → Rate at which mass collides with the target.

a) **Projectiles stop upon impact:** $\Delta v = v_f - v_i = 0 - v = -v$

b) **Projectiles bounce:** $\Delta v = v_f - v_i = -v - v = -2v$

V. Momentum and kinetic energy in collisions

Assumptions: <u>Closed systems</u> (no mass enters or leaves them)

<u>Isolated systems</u> (no external forces act on the bodies within the system)

- Elastic collision: If the total kinetic energy of the system of two colliding bodies is unchanged (conserved) by the collision.

  Example: Superball into hard floor.

- Inelastic collision: The kinetic energy of the system is not conserved → some goes into thermal energy, sound, etc.

- Completely inelastic collision: After the collision the bodies loose energy and stick together.

  Example: Ball of wet putty into floor

Conservation of linear momentum:

The total linear momentum of a closed, isolated system cannot change. (P can only be changed by external forces and the forces in the collision are internal)

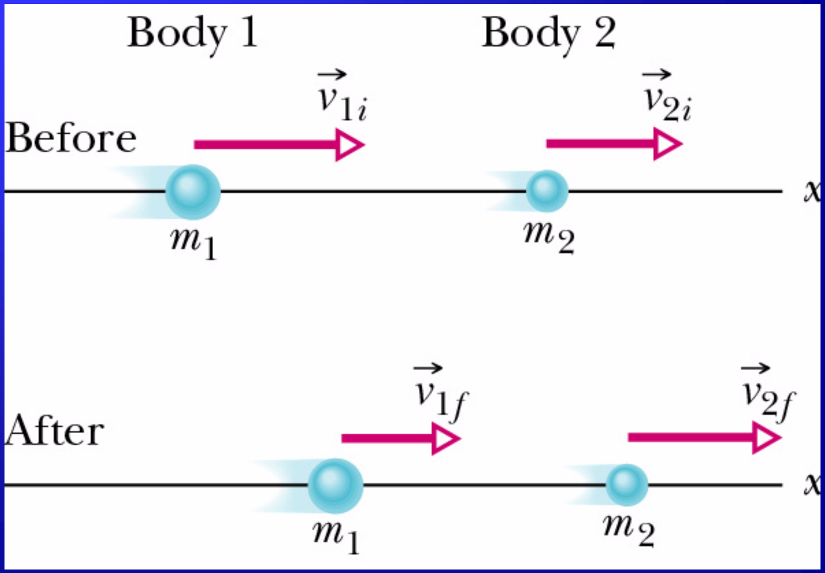
In a closed, isolated system containing a collision, the linear momentum of each colliding body may change but the total linear momentum $\vec{P}$ of the system cannot change, whether the collision is elastic or inelastic.

VI. Inelastic collisions in 1D

$$(Total\ momentum\ \vec{p}_i\ before\ collision) =$$
$$(Total\ momentum\ \vec{p}_f\ after\ collision)$$

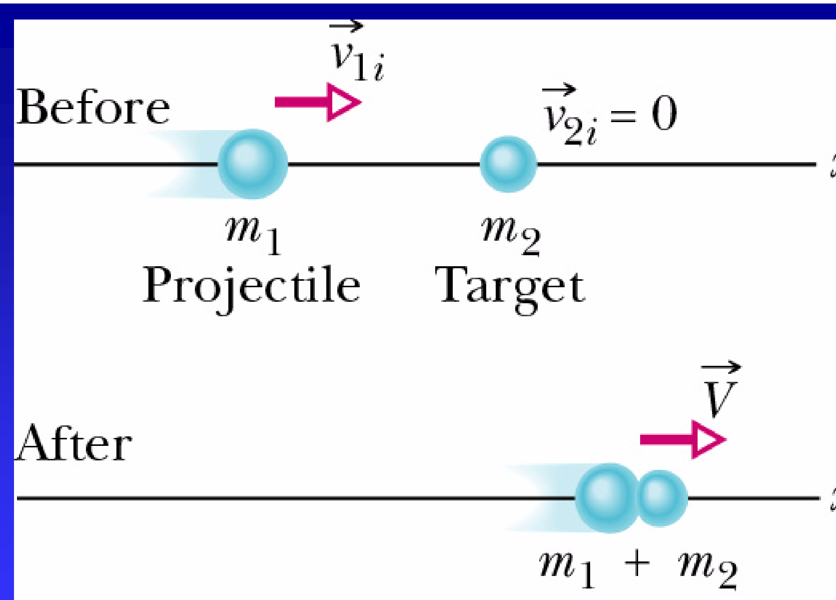
Conservation of linear momentum

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}$$

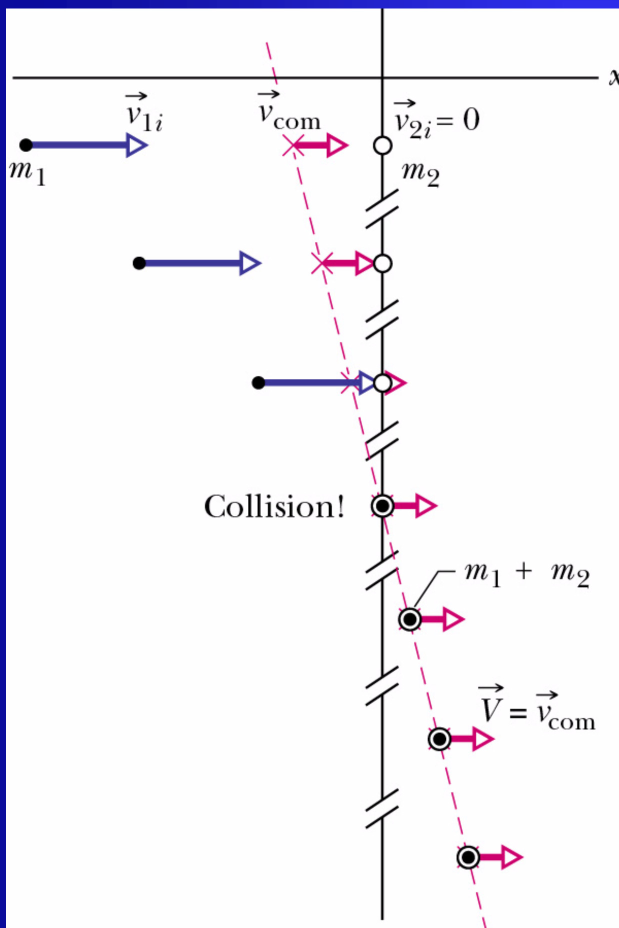


- **Completely inelastic collision:**

$$m_1 v_{1i} = (m_1 + m_2)V \rightarrow V = \frac{m_1}{m_1 + m_2} v_{1i}$$



- **Velocity of the center of mass:**



In a closed, isolated system, the velocity of COM of the system cannot be changed by a collision. (No net external force).

$$\vec{P} = M\vec{v}_{com} = (m_1 + m_2)\vec{v}_{com}$$

$$\vec{P} \text{ conserved} \rightarrow \vec{P} = \vec{p}_{1i} + \vec{p}_{2i}$$

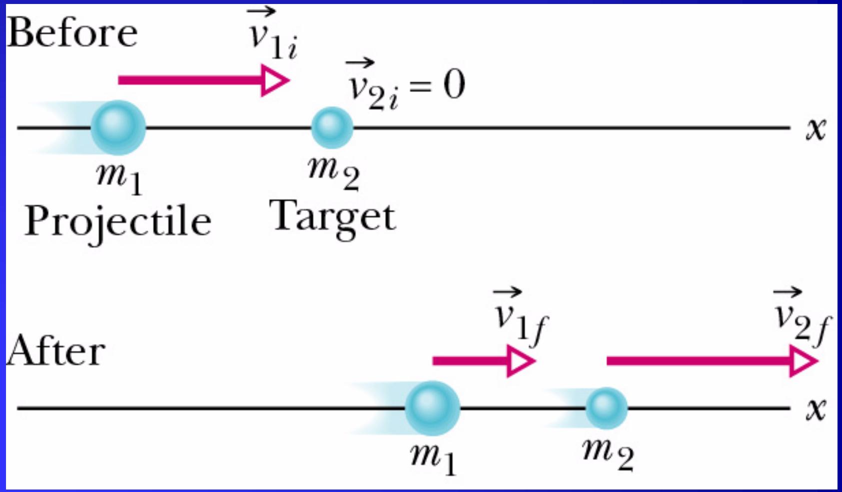
$$\rightarrow \vec{v}_{com} = \frac{\vec{P}}{m_1 + m_2} = \frac{\vec{p}_{1i} + \vec{p}_{2i}}{m_1 + m_2} = \frac{\vec{p}_{1f} + \vec{p}_{2f}}{m_1 + m_2}$$

Completely inelastic collision → $\vec{V} = \vec{v}_{com}$

VII. Elastic collisions in 1D

$$(Total\ \ kinetic\ energy\ before\ collision) = (Total\ \ kinetic\ energy\ after\ collision)$$

In an elastic collision, the kinetic energy of each colliding body may change, but the total kinetic energy of the system does not change.



- Stationary target:

Closed, isolated system →

$$m_1 v_{1i} = m_1 v_{1f} + m_2 v_{2f}$$

Linear momentum

$$\frac{1}{2} m_1 v_{i1}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$$

Kinetic energy

$$m_1 (v_{1i} - v_{1f}) = m_2 v_{2f} \qquad (1)$$

$$m_1 (v_{1i}^2 - v_{1f}^2) = m_2 v_{2f}^2 = m_1 (v_{1i} + v_{1f})(v_{1i} - v_{1f}) \qquad (2)$$

- Stationary target:

$$Dividing\ \ (2)/(1) \rightarrow v_{2f} = v_{1i} + v_{1f} \qquad (3)$$

$$From\ \ (1) \rightarrow v_{2f} = \frac{m_1}{m_2}(v_{1i} - v_{1f}) \qquad (1)\ \ in\ \ (3)\ \ v_{1f} = v_{2f} - v_{1i} = \frac{m_1}{m_2}(v_{1i} - v_{1f}) - v_{1i} \rightarrow$$

$$\boxed{v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} \qquad v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i}}$$

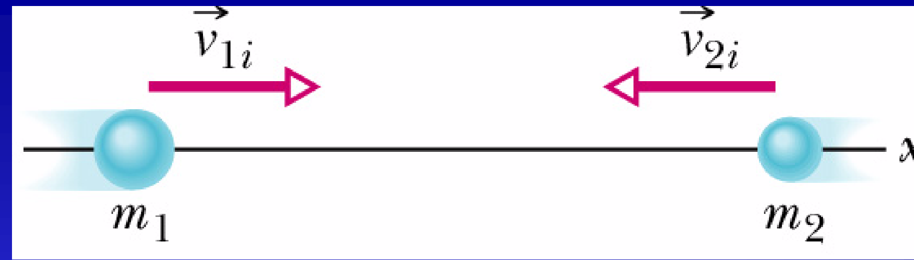
v_{2f} >0 always
v_{1f} >0 if m_1>m_2 → forward mov.
v_{1f} <0 if m_1<m_2 → rebounds

-Equal masses: $m_1=m_2$ → $v_{1f}=0$ and $v_{2f} = v_{1i}$ → In head-on collisions bodies of equal masses simply exchange velocities.

- Massive target: $m_2 >> m_1$ → $v_{1f} \approx -v_{1i}$ and $v_{2f} \approx (2m_1/m_2)v_{1i}$ → Body 1 bounces back with approximately same speed. Body 2 moves forward at low speed.

- Massive projectile: $m_1 >> m_2$ → $v_{1f} \approx v_{1i}$ and $v_{2f} \approx 2v_{1i}$ → Body 1 keeps on going scarcely slowed by the collision. Body 2 charges ahead at twice the initial speed of the projectile.

VII. Elastic collisions in 1D



- Moving target:

Closed, isolated system → $m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$ Linear momentum

$$\frac{1}{2} m_1 v_{i1}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$$ Kinetic energy

$$m_1 (v_{1i} - v_{1f}) = -m_2 (v_{2i} - v_{2f}) \qquad (1)$$

$$m_1 (v_{1i} - v_{1f})(v_{1i} + v_{1f}) = -m_2 (v_{2i} - v_{2f})(v_{2i} + v_{2f}) \qquad (2)$$

$$Dividing\ (2)/(1) + a\lg ebra \rightarrow \boxed{\begin{aligned} v_{1f} &= \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i} \\ v_{2f} &= \frac{2m_1}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2i} \end{aligned}}$$

VIII. Collisions in 2D

Closed, isolated system →

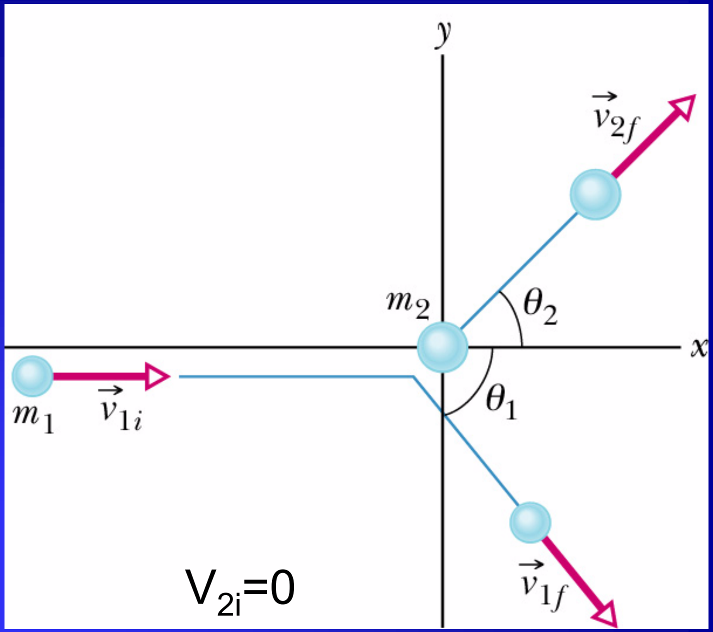
$$\vec{P}_{1i} + \vec{P}_{2i} = \vec{P}_{1f} + \vec{P}_{2f}$$

Linear momentum conserved

Elastic collision →

$$K_{1i} + K_{2i} = K_{1f} + K_{2f}$$

Kinetic energy conserved



Example:

$$x - axis \rightarrow m_1 v_{1i} = m_1 v_{1f} \cos\theta_1 + m_2 v_{2f} \cos\theta_2$$

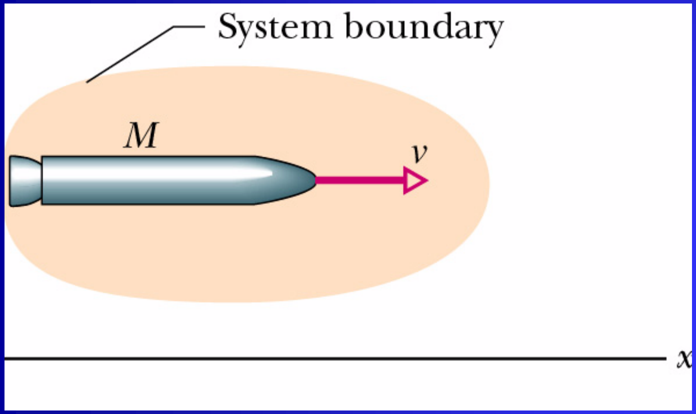
$$y - axis \rightarrow 0 = -m_1 v_{1f} \sin\theta_1 + m_2 v_{2f} \sin\theta_2$$

If the collision is elastic →

$$\frac{1}{2} m_1 v_{i1}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$$

IV. Systems with varying mass

Example: most of the mass of a rocket on its launching is fuel that gets burned during the travel.



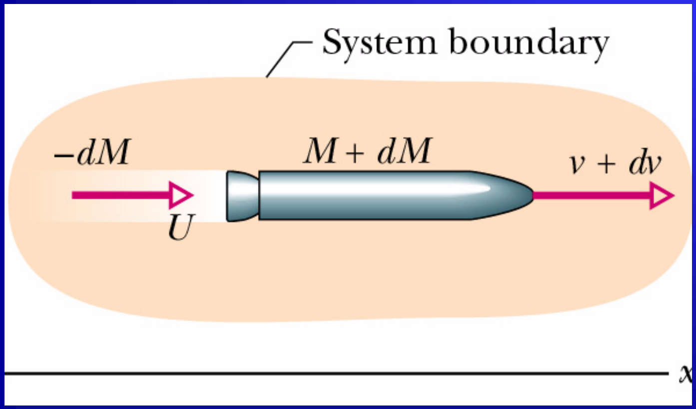
System: rocket + exhaust products

Closed and isolated → mass of this system does not change as the rocket accelerates.

$P = cte \rightarrow P_i = P_f$

After dt

dM < 0



$$Mv = -dM \cdot U + (M + dM) \cdot (v + dv)$$

Linear momentum of exhaust products released during the interval dt

Linear momentum of rocket at the end of dt

Velocity of rocket relative to frame = (velocity of rocket relative to products)+ + (velocity of products relative to frame)

$$(v+dv)=v_{rel}+U \rightarrow U=(v+dv)-v_{rel}$$

$$Mv=-dM\cdot U+(M+dM)\cdot(v+dv)$$

$$Mv=-dM[(v+dv)-v_{rel}]+(M+dM)(v+dv)$$

$$Mv=-\cancel{vdM}-\cancel{dvdM}+v_{rel}dM+Mv+Mdv+\cancel{vdM}+\cancel{dvdM}=Mv+v_{rel}dM+Mdv$$

$$Mdv=-v_{rel}dM \rightarrow -\frac{dM}{dt}v_{rel}=M\frac{dv}{dt}$$

R=Rate at which the rocket losses mass= -dM/dt = rate of fuel consumption

$$-\frac{dM}{dt}v_{rel}=M\frac{dv}{dt}\rightarrow \boxed{R\cdot v_{rel}=Ma}$$

First rocket equation

$$-\frac{dM}{dt}v_{rel} = M\frac{dv}{dt} \rightarrow dv = -\frac{dM}{M}v_{rel} \rightarrow \int_{v_i}^{v_f} dv = -v_{rel}\int_{M_i}^{M_f}\frac{dM}{M} = -v_{rel}\left(\ln M_f - \ln M_i\right)$$

$$\boxed{v_f - v_i = v_{rel}\ln\frac{M_i}{M_f}}$$

Second rocket equation