Chapter 3 – 2D and 3D Motion

I. Definitions

II. Projectile motion

III. Uniform circular motion

IV. Relative motion

I. Definitions

Position vector: extends from the origin of a coordinate system to the particle.

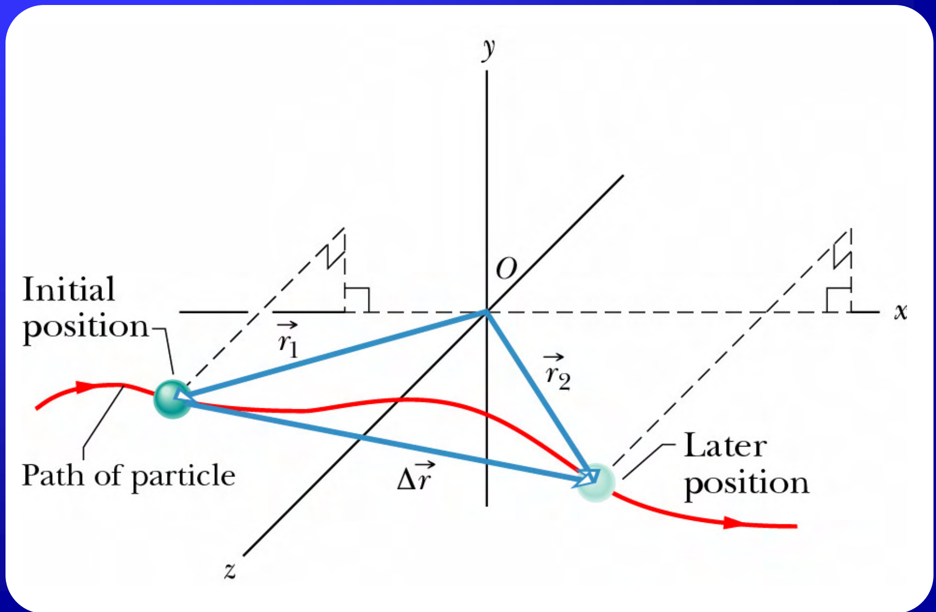
$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \qquad (4.1)$$

Displacement vector: represents a particle's position change during a certain time interval.

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1 = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k} \qquad (4.2)$$

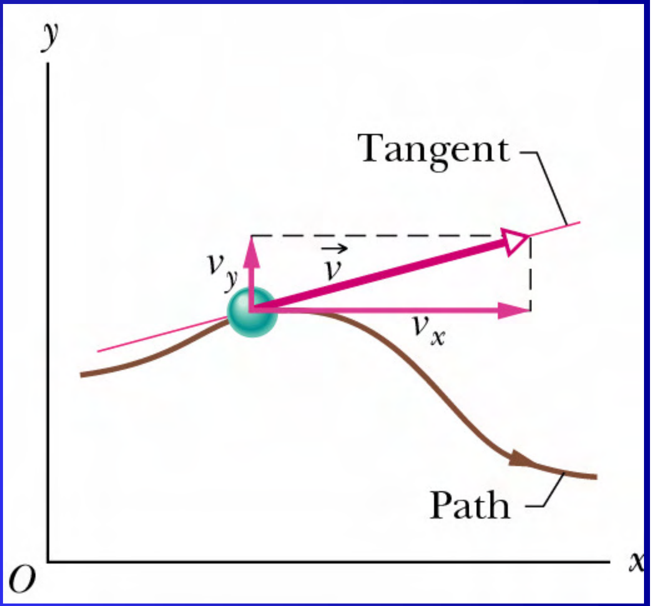
Average velocity:

$$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta x}{\Delta t}\hat{i} + \frac{\Delta y}{\Delta t}\hat{j} + \frac{\Delta z}{\Delta t}\hat{k} \qquad (4.3)$$



Instantaneous velocity:

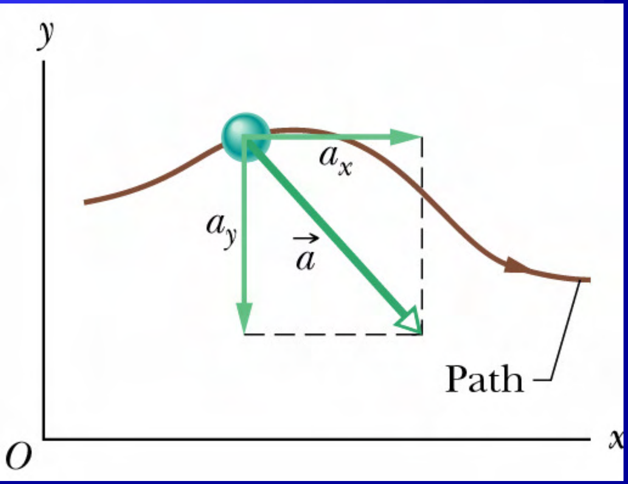
$$\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} \qquad (4.4)$$



- The direction of the instantaneous velocity of a particle is always tangent to the particle's path at the particle's position

Average acceleration:

$$\vec{a}_{avg} = \frac{\vec{v}_2 - \vec{v}_1}{\Delta t} = \frac{\Delta\vec{v}}{\Delta t} \qquad (4.5)$$



Instantaneous acceleration:

$$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k} \qquad (4.6)$$

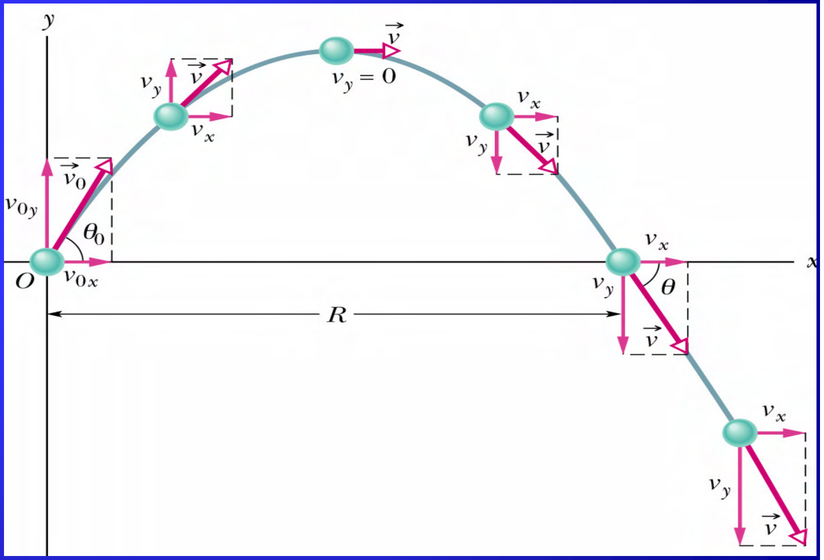
II. Projectile motion

Motion of a particle launched with initial velocity, $\vec{v}_0$ and free fall acceleration $\vec{g}$.

The horizontal and vertical motions are independent from each other.

- **Horizontal motion:** $a_x=0 \rightarrow v_x=v_{0x}=$ constant

$$x - x_0 = v_{0x}t = (v_0 \cos\theta_0)t \qquad (4.7)$$



Range (R): horizontal distance traveled by a projectile before returning to launch height.

- **Vertical motion:** $a_y= -g =$ constant

$$y - y_0 = v_{0y}t - \frac{1}{2}gt^2 = (v_0 \sin\theta_0)t - \frac{1}{2}gt^2 \qquad (4.8)$$

$$v_y = v_0 \sin\theta_0 - gt \qquad (4.9)$$

$$v_y^{\,2} = (v_0 \sin\theta_0)^2 - 2g(y - y_0) \qquad (4.10)$$

- Trajectory: projectile's path. $x_0 = y_0 = 0$

$$(4.7) + (4.8) \rightarrow t = \frac{x}{v_0 \cos\theta_0} \rightarrow y = v_0 \sin\theta_0 \frac{x}{v_0 \cos\theta_0} - \frac{1}{2} g \left(\frac{x}{v_0 \cos\theta_0} \right)^2 \rightarrow$$

$$y = (\tan\theta_0) x - \frac{g x^2}{2 (v_0 \cos\theta_0)^2} \qquad (4.11)$$

- Horizontal range: R = x-x_0; y-y_0=0.

$$R = (v_0 \cos\theta_0) t \rightarrow t = \frac{R}{v_0 \cos\theta_0}$$

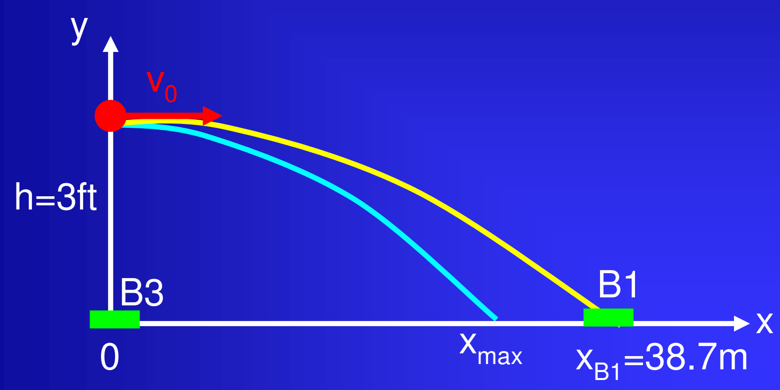
$$0 = (v_0 \sin\theta_0) t - \frac{1}{2} g t^2 = (v_0 \sin\theta_0) \frac{R}{v_0 \cos\theta_0} - \frac{1}{2} g \left(\frac{R}{v_0 \cos\theta_0} \right)^2 = R \tan\theta_0 - \frac{1}{2} g \frac{R^2}{v_0^2 \cos^2\theta_0} \rightarrow$$

$$\boxed{R = \frac{2 \sin\theta_0 \cos\theta_0}{g} v_0^2 = \frac{v_0^2}{g} \sin 2\theta_0} \qquad (4.12)$$

(Maximum for a launch angle of 45º)

Overall assumption: the air through which the projectile moves has no effect on its motion → friction neglected.

122: A third baseman wishes to throw to first base, 127 feet distant. His best throwing speed is 85 mi/h. (a) If he throws the ball horizontally 3 ft above the ground, how far from first base will it hit the ground? (b) From the same initial height, at what upward angle must the third baseman throw the ball if the first baseman is to catch it 3 ft above the ground? (c) What will be the time of flight in that case?



$$\left(85\frac{\cancel{mi}}{\cancel{h}}\right)\cdot\left(\frac{1\cancel{h}}{3600s}\right)\cdot\left(\frac{1609m}{1\cancel{mi}}\right)=38m/s$$

$$(3\,feet)\cdot\left(\frac{0.305m}{1\,foot}\right)=0.91m$$

Vertical movement

$$y-y_0=\cancel{v_{0y}}t-\frac{1}{2}gt^2$$

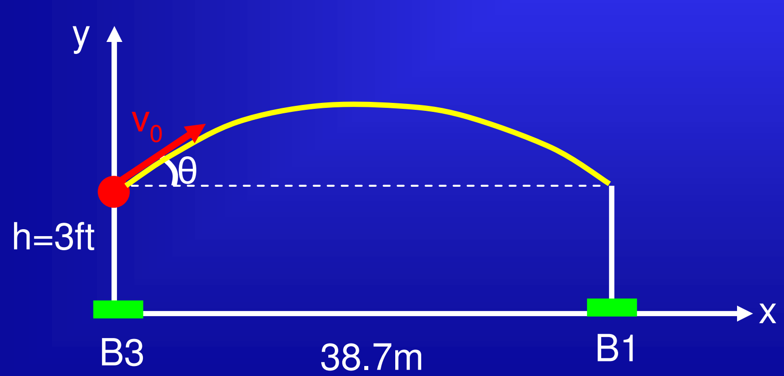
$$0-0.91m=-4.9t^2\rightarrow t=0.43s$$

Horizontal movement

$$x_{max}-x_0=v_{0x}t$$

$$x_{max}-0=38t=(38m/s)(0.43s)=16.4m\ \ from\ B3$$

The ball will hit ground at 22.3 m from B1



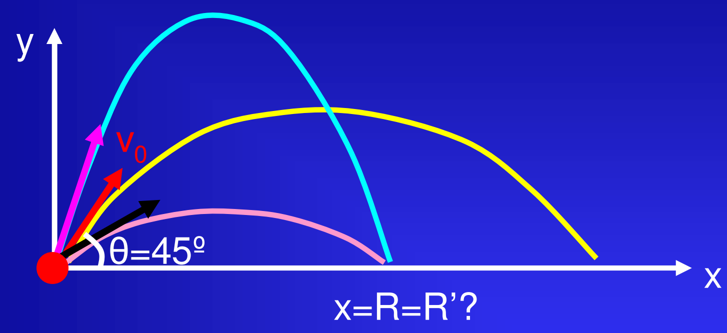
$$y-y_0=0=v_{0y}t-\frac{1}{2}gt^2\rightarrow v_{0y}=4.9t=v_0\sin\theta\rightarrow t=\frac{38\sin\theta}{4.9}$$

$$v_{0x}=\frac{38.7m}{t}=v_0\cos\theta\rightarrow t=\frac{38.7}{38\cos\theta}=\boxed{1s}$$

$$\frac{38.7}{38\cos\theta}=\frac{38\sin\theta}{4.9}\rightarrow 189.63=1444\sin\theta\cos\theta\rightarrow$$

$$0.13=0.5\sin 2\theta\rightarrow\boxed{\theta=7.6^\circ}$$

N7: In Galileo's Two New Sciences, the author states that "for elevations (angles of projection) which exceed or fall short of 45º by equal amounts, the ranges are equal…" Prove this statement.



$$\theta = 45^\circ$$
$$\theta_1 = 45^\circ + \delta\theta$$
$$\theta_2 = 45^\circ - \delta\theta$$

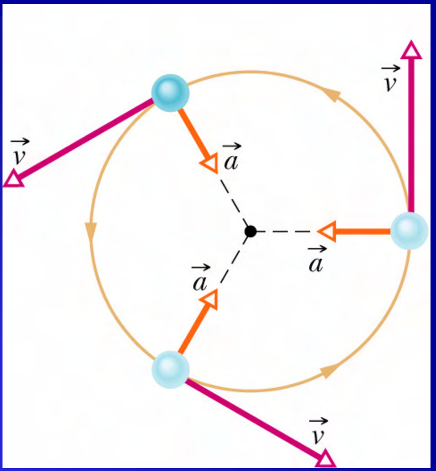
$$Range: \quad R = \frac{v_0^2}{g}\sin 2\theta_0 \rightarrow d_{\max} \ \ at \ \ h = 0$$

$$R = \frac{v_0^2}{g}\sin\left[2\left(45^\circ + \delta\theta\right)\right] = \frac{v_0^2}{g}\sin\left(90^\circ + 2\delta\theta\right)$$
$$R' = \frac{v_0^2}{g}\sin\left[2\left(45^\circ - \delta\theta\right)\right] = \frac{v_0^2}{g}\sin\left(90^\circ - 2\delta\theta\right)$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$
$$\sin(a-b) = \sin a \cos b - \cos a \sin b$$

$$R = \frac{v_0^2}{g}\left[\sin 90^\circ \cos(2\delta\theta) + \cancel{\cos 90^\circ} \sin(2\delta\theta)\right] = \boxed{\frac{v_0^2}{g}\cos(2\delta\theta)}$$
$$R' = \frac{v_0^2}{g}\left[\sin 90^\circ \cos(2\delta\theta) - \cancel{\cos 90^\circ} \sin(2\delta\theta)\right] = \frac{v_0^2}{g}\cos(2\delta\theta)$$

III. Uniform circular motion



Motion around a circle at <u>constant speed</u>.

Magnitude of velocity and acceleration constant.
Direction varies continuously.

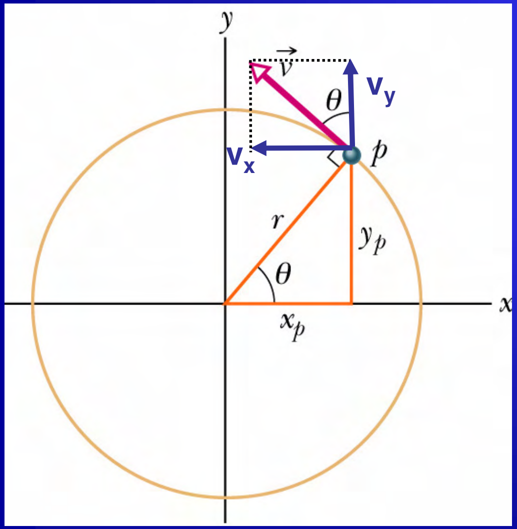
-Velocity: tangent to circle in the direction of motion.

- Acceleration: centripetal

$$a = \frac{v^2}{r} \qquad (4.13)$$

- Period of revolution:

$$T = \frac{2\pi\, r}{v} \qquad (4.14)$$



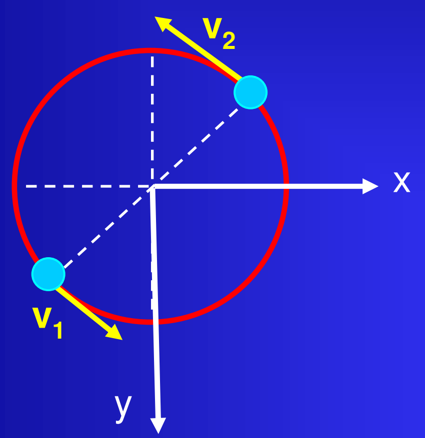
$$\vec{v} = v_x\hat{i} + v_y\hat{j} = (-v\sin\theta)\hat{i} + (v\cos\theta)\hat{j} = \left(\frac{-v\cdot y_p}{r}\right)\hat{i} + \left(\frac{v\cdot x_p}{r}\right)\hat{j}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \left(\frac{-v}{r}\frac{dy_p}{dt}\right)\hat{i} + \left(\frac{v}{r}\frac{dx_p}{dt}\right)\hat{j} = \left(\frac{-v}{r}v_y\right)\hat{i} + \left(\frac{v}{r}v_x\right)\hat{j} = \left(\frac{-v^2}{r}\cos\theta\right)\hat{i} + \left(\frac{-v^2}{r}\sin\theta\right)\hat{j}$$

$$a = \sqrt{a_x^2 + a_y^2} = \frac{v^2}{r}\sqrt{\cos^2\theta + \sin^2\theta} = \frac{v^2}{r}$$

$$\vec{a}\ directed\ along\ radius \rightarrow \tan\phi = \frac{a_y}{a_x} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

54. A cat rides a merry-go-round while turning with uniform circular motion. At time t_1= 2s, the cat's velocity is: $\vec{v}_1$= (3m/s)$\hat{i}$+(4m/s)$\hat{j}$, measured on an horizontal xy coordinate system. At time t_2=5s its velocity is: $\vec{v}_2$= (-3m/s)$\hat{i}$+(-4m/s)$\hat{j}$. What are (a) the magnitude of the cat's centripetal acceleration and (b) the cat's average acceleration during the time interval t_2-t_1?



In 3s the velocity is reversed → the cat reaches the opposite side of the circle

$$v = \sqrt{3^2 + 4^2} = 5m/s$$

$$T = \frac{2\pi r}{v} \rightarrow 3s = \frac{\pi r}{5m/s} \rightarrow r = 4.77m$$

$$a_c = \frac{v^2}{r} = \frac{25m^2/s^2}{4.77m} = 5.23m/s^2$$

$$\vec{a}_{avg} = \frac{\vec{v}_2 - \vec{v}_1}{\Delta t} = \frac{(-6m/s)\hat{i} - (8m/s)\hat{j}}{3s} = (-2m/s^2)\hat{i} - (2.67m/s^2)\hat{j}$$

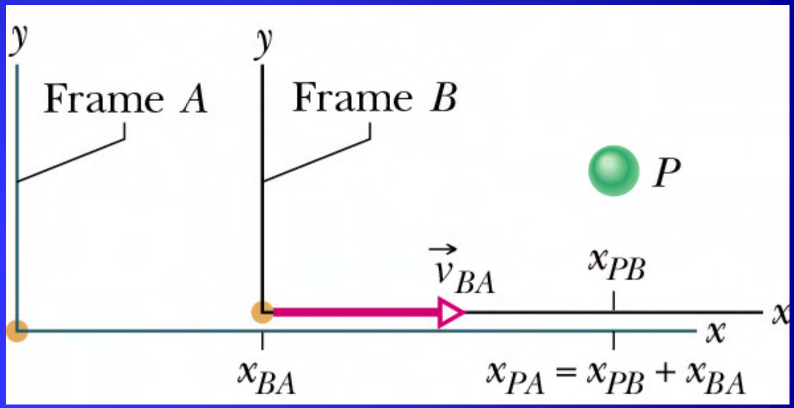
$$\left|\vec{a}_{avg}\right| = 3.33m/s^2$$

IV. Relative motion

Particle's velocity depends on reference frame

$$v_{PA} = v_{PB} + v_{BA} \qquad (4.15)$$

1D



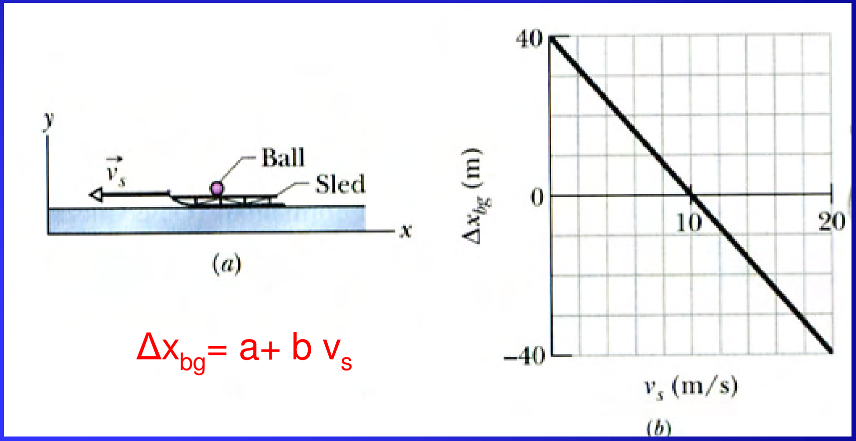
Frame moves at constant velocity

$$\frac{d}{dt}(v_{PA}) = \frac{d}{dt}(v_{PB}) + \cancel{\frac{d}{dt}(v_{BA})}^{0} \rightarrow a_{PA} = a_{PB} \qquad (4.16)$$

Observers on different frames of reference measure the same acceleration for a moving particle if their relative velocity is constant.

75. A sled moves in the negative x direction at speed v_s while a ball of ice is shot from the sled with a velocity $\vec{v}_0 = v_{0x}\hat{i} + v_{0y}\hat{j}$ relative to the sled. When the ball lands, its horizontal displacement Δx_{bg} relative to the ground (from its launch position to its landing position) is measured. The figure gives Δx_{bg} as a function of v_s. Assume it lands at approximately its launch height. What are the values of (a) v_{0x} and (b) v_{0y}?
The ball's displacement Δx_{bs} relative to the sled can also be measured. Assume that the sled's velocity is not changed when the ball is shot. What is Δx_{bs} when v_s is (c) 5m/s and (d) 15m/s?

$$\vec{v}_s = -v_s\hat{i}$$
$$\vec{v}_{0,rel} = v_{0x}\hat{i} + v_{0y}\hat{j}$$
$$\vec{v}_{0g} = (v_{0x} - v_s)\hat{i} + v_{0y}\hat{j} \longrightarrow$$ Launch velocity relative to ground



$\Delta x_{bg} = a + b\, v_s$

Displacements relative to ground

$$x_{land} - x_{launch} = \Delta x_{bg} = (v_{0x} - v_s)t_{flight}$$
$$y_{land} - y_{launch} = 0 = v_{0y}t_{flight} - 0.5gt_{flight}^2$$

$$0 = v_{0y}t_{flight} - 0.5gt_{flight}^2 \rightarrow t = \frac{2v_{0y}}{g}$$

$$\Delta x_{bg} = (v_{0x} - v_s)\frac{2v_{0y}}{g} = \underbrace{\frac{2v_{0x}v_{0y}}{g}}_{a} - \underbrace{\frac{2v_{0y}}{g}}_{b}v_s$$

$$-\frac{2v_{0y}}{g} = -\frac{80m}{20m/s} \rightarrow v_{0y} = 19.6m/s$$

$$\Delta x_{bg} = 0 \rightarrow v_s = 10m/s \rightarrow 0 = \frac{2v_{0x}}{g} - \frac{2\cdot(10m/s)}{g} \rightarrow v_{0x} = 10m/s$$

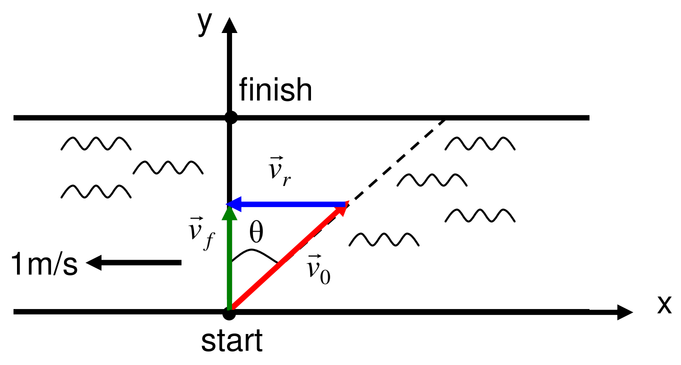
$$t_{flight} = \frac{2v_{0y}}{g} = 4s \qquad v_0 = \sqrt{10^2 + 19.6^2} = 22m/s$$

Displacements relative to the sled

$$\Delta x_{bs} = v_{0x}t_{flight} = (10m/s)\cdot 4s = 40m$$

Relative to the sled, the displacement does not depend on the sled's speed →Answer (c)= Answer (d)

(iii) A dog wishes to cross a river to a point directly opposite as shown. It can swim at 2m/s in still water and the river is flowing at 1m/s. At what angle with respect to the line joining the starting and finishing points should it start swimming?



$$\vec{v}_r = (-1m/s)\hat{i}$$

$$\vec{v}_0 = 2\sin\theta\ \hat{i} + 2\cos\theta\ \hat{j}$$

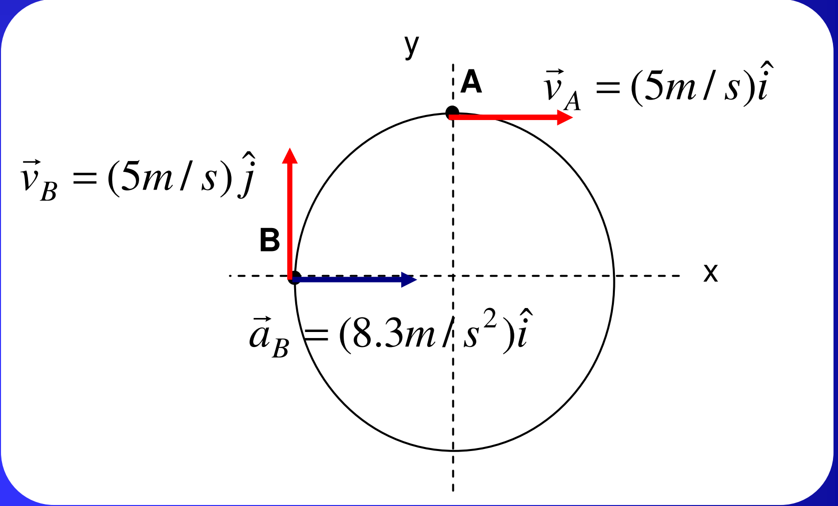
$$\vec{v}_f = \vec{v}_r + \vec{v}_0 = (-\hat{i} + 2\sin\theta\ \hat{i} + 2\cos\theta\ \hat{j})m/s = (v_f\,\hat{j})m/s$$

$$2\sin\theta - 1 = 0 \rightarrow \theta = 30^\circ$$

$$2\cos\theta = v_f \rightarrow v_f = \sqrt{3}m/s$$

(ii) A particle moves with constant speed around the circle below. When it is at point A its coordinates are x=0, y=3m and its velocity is (5m/s)$\hat{i}$. What are its velocity and acceleration at point B? Express your answer in terms of unit vectors.

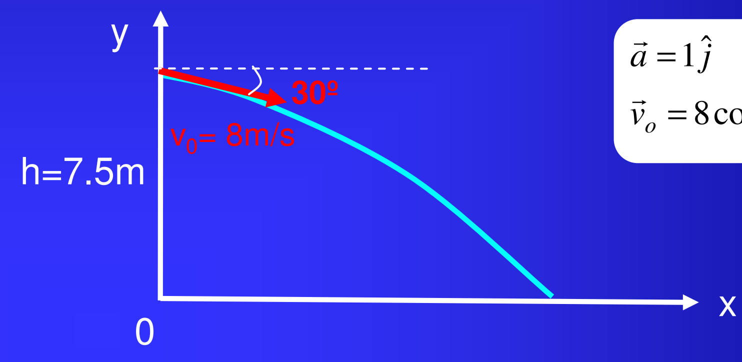
$$\vec{a}_B = \frac{v^2}{r}\hat{i} = \frac{25m^2/s^2}{3m}\hat{i} = (8.3m/s^2)\hat{i}$$



120. A hang glider is 7.5 m above ground level with a velocity of 8m/s at an angle of 30º below the horizontal and a constant acceleration of 1m/s², up. (a) Assume t=0 at the instant just described and write an equation for the elevation y of the hang glider as a function of t, with y=0 at ground level. (b) Use the equation to determine the value of t when y=0. (c) Explain why there are two solutions to part B. Which one represents the time it takes the hang glider to reach ground level? (d) how far does the hang glider travel horizontally during the interval between t=0 and the time it reaches the ground? For the same initial position and velocity, what constant acceleration will cause the hang glider to reach ground level with zero velocity? Express your answer in terms of unit vectors.

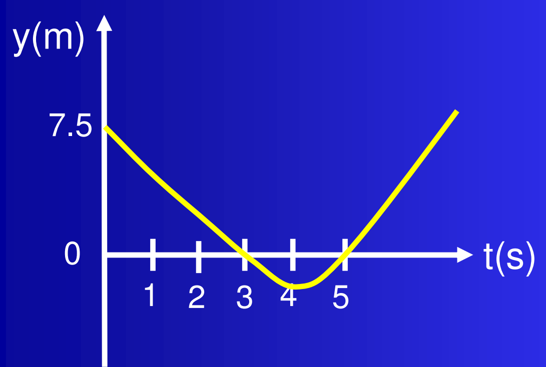
$$y - y_0 = v_{0y}t + \frac{1}{2}at^2 \rightarrow y = 7.5 - 4t + 0.5t^2$$

$$y = 0 \rightarrow 0 = 7.5 - 4t + 0.5t^2 \rightarrow t_1 = 5s, \; t_2 = 3s$$



$$\vec{a} = 1\hat{j}$$

$$\vec{v}_o = 8\cos 30^\circ \hat{i} - 8\sin 30^\circ \hat{j}$$



If the ground was not solid, the glider would swoop down, passing through the surface, then back up again, with the two times of passing being t=3s, t=5s.

$$d_{\max} = v_{0x}t = (6.93m/s)\cdot(3s) = 20.78m$$

$$y = 0 \rightarrow \vec{v}_f = v_{fx}\hat{i} + v_{fy}\hat{j} = \vec{0}$$

Vertical movement:

$$v_y{}^2 = v_{0y}^2 + 2a_y\cdot(y - y_0) = 4^2 - 15a_y \rightarrow a_y = 1.1m/s^2$$

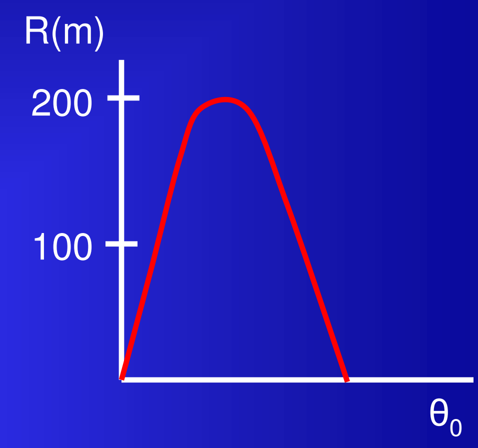
$$v_y = v_{0y} + a_y t = 0 \rightarrow 0 = -4 + 1.1t \rightarrow t = 3.75s$$

Horizontal movement:

$$0 = v_x = v_{0x} + a_x\cdot t = 6.93 + 3.75a_x \rightarrow a_x = -1.85m/s^2$$

$$\vec{a} = (-1.85m/s^2)\hat{i} + (1.1m/s^2)\hat{j}$$

40. A ball is to be shot from level ground with certain speed. The figure below shows the range R it will have versus the launch angle θ_0 at which it can be launched. The choice of θ_0 determines the flight time; let t_{max} represent the maximum flight time. What is the least speed the ball will have during its flight if θ_0 is chosen such as that the flight time is $0.5t_{max}$?



$$y - y_0 = 0 = v_{0y}t - 0.5gt^2 = v_0 \sin\theta_0 t - 4.9t^2$$

$$\rightarrow t = \frac{2v_0 \sin\theta_0}{g} \rightarrow \sin\theta_0 = 1 \rightarrow t_{\max} = \frac{2v_0}{g}$$

$$t_{flight} = \frac{2v_0 \sin\theta_0}{g} = 0.5t_{\max} \rightarrow \frac{2v_0 \sin\theta_0}{g} = \frac{v_0}{g} \rightarrow \theta_0 = \sin^{-1} 0.5 = 30^\circ$$

The lowest speed occurs at the top of the trajectory (half of total time of flight), when the velocity has simply an x-component.

$$v_{\min} = v_x \ \textit{at half trajectory} = v_{0x} = v_0 \cos 30^\circ \ \textit{for } 0.5t_{\max}$$

$$\textit{Graph} \quad R_{\max} = 240m \ \textit{for } \theta_0 = 45^\circ$$

$$R = \frac{v_0^2}{g}\sin 2\theta_0 \rightarrow 240m = \frac{v_0^2}{g}\sin 90^\circ = \frac{v_0^2}{g} \rightarrow v_0 = 48.5m/s$$

$$v_{\min} = v_x = (48.5m/s)\cos 30^\circ = 42m/s$$