Chapter 3
Kinematics in two dimensions

Goals for Chapter 3

• to study position, velocity, and acceleration vectors in two dimensions
• to understand how displacement, velocity, and acceleration are applied in two dimensional motion
• to study two-dimensional motion as it occurs in the motion of projectiles
• to study the concept of relative motion

Most important concept in two-dimensional motion

• two-dimensional motion can be decomposed into motion in x-direction and motion in y-direction
  “x” and “y” components of motion are independent
  • Person on a cart throws ball vertically upwards
    – Seen from two different frames of reference

Reference frame on moving cart
Reference frame on the ground

3.1 Displacement, velocity, acceleration

\[ \vec{r}_0 = \text{initial position} \]
\[ \vec{r} = \text{final position} \]
\[ \Delta \vec{r} = \vec{r} - \vec{r}_0 = \text{displacement} \]

Average velocity is the displacement divided by the elapsed time.

\[ \bar{v} = \frac{\vec{r} - \vec{r}_0}{t - t_0} = \frac{\Delta \vec{r}}{\Delta t} \]

2-d Displacement, velocity, acceleration

The instantaneous velocity indicates how fast the car moves and the direction of motion at each instant of time.

\[ \bar{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} \]

2-d Displacement, velocity, acceleration

Definition of average acceleration

\[ \overline{\vec{a}} = \frac{\bar{v} - \bar{v}_0}{t - t_0} = \frac{\Delta \bar{v}}{\Delta t} \]

The instantaneous velocity vector \( \vec{v} \) is always tangent to the x-y path.

Definition of instantaneous acceleration

\[ \vec{a} = \lim_{\Delta \vec{v} \to 0} \frac{\Delta \vec{v}}{\Delta t} \]
Motion in two dimensions

Equations of kinematics in two dimensions

Projectile motion

Acceleration must now be considered during change in magnitude AND/OR direction.

Please note that the above example is not a "projectile" motion near the surface of earth.

Equations of kinematics for constant acceleration

<table>
<thead>
<tr>
<th>Variables related</th>
<th>Equation</th>
<th>( a = \text{const} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>velocity, time, acceleration</td>
<td>( v = v_0 + at )</td>
<td></td>
</tr>
<tr>
<td>position, velocity, time</td>
<td>( x = \frac{1}{2} \left( v_0 + v \right) t )</td>
<td></td>
</tr>
<tr>
<td>initial position ( x_0 = 0 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>velocity, position, acceleration</td>
<td>( v^2 = v_0^2 + 2ax )</td>
<td></td>
</tr>
<tr>
<td>position, time, acceleration</td>
<td>( x = v_0 t + \frac{1}{2} at^2 )</td>
<td></td>
</tr>
</tbody>
</table>

The independence of \( x \) and \( y \) motion

Notice that the vertical motion under free fall spaces out exactly as the vertical motion of the projectile.

The \( x \) part of the motion occurs exactly as it would if the \( y \) part did not occur at all, and vice versa.

3.2 Equations of kinematics in two dimensions

\( x \)-component (horizontal)

\( v_x = v_{ox} + a_x t \) \hspace{1cm} \( x = \frac{1}{2} \left( v_{ox} + v_x \right) t \)

\( x = v_{ox} t + \frac{1}{2} a_x t^2 \) \hspace{1cm} \( v_x^2 = v_{ox}^2 + 2a_x x \)

\( y \)-component (vertical)

\( v_y = v_{oy} + a_y t \) \hspace{1cm} \( y = v_{oy} t + \frac{1}{2} a_y t^2 \)

\( y = \frac{1}{2} \left( v_{oy} + v_y \right) t \) \hspace{1cm} \( v_y^2 = v_{oy}^2 + 2a_y y \)
Example: A moving spacecraft

In the \( x \) direction, the spacecraft has an initial velocity component of +22 m/s and an acceleration of +24 m/s\(^2\). In the \( y \) direction, the analogous quantities are +14 m/s and an acceleration of +12 m/s\(^2\). Find (a) \( x \) and \( v_x \), (b) \( y \) and \( v_y \), and (c) the final velocity of the spacecraft at time 7.0 s.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( a_x )</th>
<th>( v_x )</th>
<th>( v_{ox} )</th>
<th>( t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>?</td>
<td>+24.0 m/s(^2)</td>
<td>?</td>
<td>+22 m/s</td>
<td>7.0 s</td>
</tr>
</tbody>
</table>

\[ x = v_{ox}t + \frac{1}{2}a_xt^2 \]

\[ = (22 \text{ m/s})(7.0 \text{ s}) + \frac{1}{2}(24 \text{ m/s}^2)(7.0 \text{ s})^2 = +740 \text{ m} \]

\[ v_x = v_{ox} + a_xt \]

\[ = (22 \text{ m/s}) + (24 \text{ m/s}^2)(7.0 \text{ s}) = +190 \text{ m/s} \]

<table>
<thead>
<tr>
<th>( y )</th>
<th>( a_y )</th>
<th>( v_y )</th>
<th>( v_{oy} )</th>
<th>( t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>?</td>
<td>+12.0 m/s(^2)</td>
<td>?</td>
<td>+14 m/s</td>
<td>7.0 s</td>
</tr>
</tbody>
</table>

\[ y = v_{oy}t + \frac{1}{2}a_yt^2 \]

\[ = (14 \text{ m/s})(7.0 \text{ s}) + \frac{1}{2}(12 \text{ m/s}^2)(7.0 \text{ s})^2 = +390 \text{ m} \]

\[ v_y = v_{oy} + a_yt \]

\[ = (14 \text{ m/s}) + (12 \text{ m/s}^2)(7.0 \text{ s}) = +98 \text{ m/s} \]

3.3 Projectile motion: Conceptual example

Suppose you are driving a convertible with the top down. The car is moving to the right at constant velocity. You point a rifle straight up into the air and fire it. In the absence of air resistance, where would the bullet land – behind you, ahead of you, or in the barrel of the rifle?

\[ v = \sqrt{(190 \text{ m/s})^2 + (98 \text{ m/s})^2} = 210 \text{ m/s} \]

\[ \theta = \tan^{-1}(98/190) = 27^\circ \]
3.3 Projectile Motion

Determined by the initial velocity, gravity, (air resistance ignored)

Footballs, baseballs, ... any projectiles will follow this parabolic path

\[ \vec{a}_y = -g \]

- A projectile moves in a vertical plane that contains the initial velocity vector \( \vec{v}_0 \).
- Its trajectory depends only on \( \vec{v}_0 \) and \( \vec{a}_y \).

In studying projectile motion we make the following assumptions:

1. Air resistance is ignored.
2. The acceleration of gravity is constant, downward, and has a magnitude equal to \( g = 9.8 \text{ m/s}^2 \).
3. The Earth’s rotation is ignored.
4. The Earth’s curvature is ignored.

Example 3: A falling care package

The airplane is moving horizontally with a constant velocity of +115 m/s at an altitude of 1050m. Determine the time required for the care package to hit the ground.

Example 4: The velocity of the care package

What are the magnitude and direction of the final velocity of the care package?

\[
y = v_{oy} t + \frac{1}{2} a_y t^2 \quad \Rightarrow \quad y = \frac{1}{2} a_y t^2
\]

\[
t = \sqrt{\frac{2y}{a_y}} = \sqrt{\frac{2(-1050 \text{ m})}{-9.8 \text{ m/s}^2}} = 14.6 \text{ s}
\]
Example: The velocity of the care package

<table>
<thead>
<tr>
<th>$y$</th>
<th>$a_y$</th>
<th>$v_y$</th>
<th>$v_{oy}$</th>
<th>$t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1050 m</td>
<td>-9.80 m/s²</td>
<td>?</td>
<td>0 m/s</td>
<td>14.6 s</td>
</tr>
</tbody>
</table>

$$v_y = v_{oy} + a_y t = 0 + (-9.80 \text{ m/s}^2)(14.6 \text{ s}) = -143 \text{ m/s}$$

$$v = \sqrt{v_x^2 + v_y^2} = 183.5 \text{ m/s}$$

Example: The height of a kickoff

A placekicker kicks a football at an angle of 40.0 degrees and the initial speed of the ball is 22 m/s. Ignoring air resistance, determine the maximum height that the ball attains.

$$v_{ox} = v_x \cos \theta = (22 \text{ m/s}) \cos 40^\circ = 17 \text{ m/s}$$

$$v_{oy} = v_y \sin \theta = (22 \text{ m/s}) \sin 40^\circ = 14 \text{ m/s}$$

Shooting the Monkey
(tranquilizer gun)

Where does the zookeeper aim if he wants to hit the monkey? (He knows the monkey will let go as soon as he shoots!)

If there were no gravity, simply aim at the monkey

If there were no gravity, simply aim at the monkey

$r = r_0$

With gravity, still aim at the monkey!

$r = r_0 - \frac{1}{2} g t^2$

Dart hits the monkey!

Recap:
Shooting the monkey...

$x = x_0$

$y = -\frac{1}{2} g t^2$

This may be easier to visualize. It's exactly the same idea!

$x = v_x t$

$y = -\frac{1}{2} g t^2$
Example 7: The time of flight of a kickoff
What is the time of flight between kickoff and landing?

<table>
<thead>
<tr>
<th>y</th>
<th>a_y</th>
<th>v_y</th>
<th>v_o_y</th>
<th>t</th>
</tr>
</thead>
<tbody>
<tr>
<td>?</td>
<td>-9.80 m/s^2</td>
<td>0</td>
<td>14 m/s</td>
<td></td>
</tr>
</tbody>
</table>

\[ v_y^2 = v_{o_y}^2 + 2a_y y \]
\[ y = \frac{v_y^2 - v_{o_y}^2}{2a_y} \]
\[ y = \frac{0 - (14 \text{ m/s})^2}{2(-9.8 \text{ m/s}^2)} = +10 \text{ m} \]

Example 8: The range of a kickoff
Calculate the range R of the projectile.

<table>
<thead>
<tr>
<th>y</th>
<th>a_y</th>
<th>v_y</th>
<th>v_o_y</th>
<th>t</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-9.80 m/s^2</td>
<td>14 m/s</td>
<td>?</td>
<td></td>
</tr>
</tbody>
</table>

\[ y = v_{o_y}t + \frac{1}{2}a_yt^2 \]
\[ 0 = (14 \text{ m/s})t + \frac{1}{2}(-9.8 \text{ m/s}^2)t^2 \]
\[ 0 = 2(14 \text{ m/s})+(-9.8 \text{ m/s}^2)t \]
\[ t = 2.9 \text{ s} \]

\[ x = v_{o_x}t + \frac{1}{2}a_xt^2 = v_{o_x}t \]
\[ = (17 \text{ m/s})(2.9 \text{ s}) = +49 \text{ m} \]

Projectile motion - trajectory

1': \( a_x = 0; \quad a_y = -g \)
2': \( v_x = v_{o_x}; \quad v_y = -gt + v_{o_y} \)
3': \( x = v_{o_x}t + x_0 \quad \Rightarrow y = y_0 + v_{o_y}t - \frac{1}{2}gt^2 \)
\[ \Rightarrow t = (x - x_0) / v_{o_x} \quad \text{Eliminate } t \]

Parabola in x-y plane
\[ y = y_0 + \frac{v_{o_y}}{v_{o_x}}(x - x_0) - \frac{1}{2}g \left( \frac{x - x_0}{v_{o_x}} \right)^2 \]
Initial values: \( x_0 = 0, v_y = 0 \)

Trajectory \[ y = \tan \phi_0 \cdot x - \frac{1}{2} \frac{g}{v_{o_x}^2} \frac{x^2}{\cos^2 \phi_0} \]
3.3 Projectile Motion

Conceptual Example 10  Two Ways to Throw a Stone

From the top of a cliff, a person throws two stones. The stones have identical initial speeds, but stone 1 is thrown downward at some angle above the horizontal and stone 2 is thrown at the same angle below the horizontal. Neglecting air resistance, which stone, if either, strikes the water with greater velocity?

Relative velocity - a matter of the reference frame

\[ \vec{V}_{pg} = \vec{V}_{pt} + \vec{V}_{tg} \]

Relative velocity in 2-dim

The concept of relative velocity can be extended from 1-dim to 2-dim.

Velocities can carry multiple values depending on the position and motion of the object and observer.

Example: Crossing a river

A boat is crossing a river that is 1800m wide. The velocity of the boat relative to the water is 4.0m/s directed perpendicular to the current. The velocity of the water relative to the shore is 2.0m/s.

(a) What is the velocity of the boat relative to the shore?

(b) How long does it take for the boat to cross the river?
Example: An airplane in a crosswind

The compass of an airplane indicates that it is headed due north, and the airspeed indicator shows that the plane is moving through the air at 240 km/h. If there is a wind of 100 km/h from west to east, what is the velocity of the aircraft relative to the earth?

\[ \vec{V}_{PE} = \vec{V}_{PA} + \vec{V}_{AE} \]

\[ V_{PE} = \sqrt{(240)^2 + (100)^2} = 260 \text{ km/h} \]

\[ \alpha = \tan^{-1} \left( \frac{100}{240} \right) = 23^\circ \]